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ABSTRACT

The purpose of this text is to teach learning and understanding of mathematics at grades seven through nine through the use of science experiments. Previous knowledge of science on the part of students or teachers is not necessary. The text is designed to be usable with any mathematics textbook in common use. The material can be covered in three or four weeks. Chapters in the text include: (1) Introduction to Measurement; (2) Length and the Number Line; (3) Relations, Functions, and Graphing; and (4) The Linear Function. Included in the book is a glossary of terms. (RH)

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SCHOOL
MATHEMATICS
STUDY GROUP

MATHEMATICS THROUGH SCIENCE

PART I: MEASUREMENT AND GRAPHING

STUDENT TEXT

(revised edition)

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MATHEMATICS THROUGH SCIENCE

Part I: Measurement and Graphing

Student Text

(revised edition)

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PREFACE

Most of the mathematical techniques that are in use today were developed to meet practical needs. The elementary arithmetic operations have obvious uses in everyday life, but the mathematical concepts which are introduced at the junior high school level and above are not as obviously useful.

The School Mathematics Study Group has been exploring the possibility of introducing some of the basic concepts of mathematics through the use of some simple science experiments. Several units were prepared during the summer of 1963 and were used on an experimental basis in a number of classrooms during the following year. On the basis of the results of these trials, these units were revised during the summer of 1964.

This text is designed to be usable with any mathematics textbook in common use. It is not meant to replace the textbook for the course, but to supplement it. Previous acquaintance with science on the part of the student is unnecessary. The scientific principles involved are fairly simple and are explained as much as is necessary in the text. Each experiment opens a door into a new domain in mathematics: measurement, inequalities, the number line, relations and graphs. We hope that student learning and understanding will be improved through the use of this material.

The experiments have all been done in actual classroom situations. Every effort has been made to make the directions for the experiments as clear and simple as possible. The apparatus has been kept to a minimum.

The writers sincerely hope that this approach to mathematics will prove both useful and interesting to the student.

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Chapter 1

INTRODUCTION TO MEASUREMENT

1.1 Introduction

Science is the study of things and events in the world around us. The scientist takes it as his job to find order in what he observes.

Everyone knows that if a ball is thrown upward, it will fall back to the ground. The harder the ball is thrown, the longer it takes to return. If we could throw hard enough, could we make the ball disappear from the earth forever? If so, how fast would it have to go as it left our hand? These questions are very hard to answer unless, like a scientist, we have learned something about the orderly behavior of falling objects and about gravity, the downward pull of the earth. Then, we might answer the two questions as follows:

> "Yes, the ball can be made to leave the earth forever, provided its speed is at least 25,000 miles per hour as it leaves our hand."

Scientific knowledge is very highly organized knowledge compared to the knowledge most people have, and it is much more useful in predicting what may happen in the future. In this book you are going to study some of the methods that a scientist uses to organize his thinking. Some of the methods will lead you to new concepts of mathematics. You will learn how these ideas can aid the scientist in explaining the world around us.

1.2 Measures and Units

Answering the question "How much?" often leads to better understanding than answering the question "Of what sort?". As you might guess, scientists find it helpful to use numbers if they are going to use mathematics to guide their thought.

Where does the scientist get the numbers he needs? The answer, of course, is that he makes measurements. You already know something about measurement from having done a lot of it. When you weigh yourself, split a candy bar with a friend, work at a track meet, or get directions, you use measurement. You can probably give many other examples of the use of measurement in everyday life. But exactly what is measurement?

In the first place, measurement is a process for assigning numbers of units to objects or events. Different kinds of measurement require different processes. For instance, to find the volume of a fish tank we might empty quart bottles full of water into the tank until the tank is full. If fifteen bottles of water completely fill the tank, we would say that the volume is 15 quarts. The number assigned would be 15 in this case. Such a number is called a measure of the volume of the tank, and in this case we could call it the quart measure of the volume of the tank.

Another way to find the volume would be to fill up pint bottles and count the number needed to fill the tank. We would find that the volume is 30 pints. In this case the measure (or the pint measure, if you wish to be precise) is 30.

No matter how we do it, the volume stays the same, although the measures differ. Another way to say the same thing is this: 15 quarts and 30 pints are each names for the same volume. In the same way, 5 centuries and 500 years are names for the same time interval; 30 miles per hour and 44 feet per second are names for the same speed.

Notice that a measurement cannot be given by a number alone. To claim that the speed of a crawling ant is 200 doesn't say anything at all, while 200 millimeters per minute would make sense. To say that the volume of the fish tank was 15 leaves us completely in the dark until extra information is added.

What information besides the number do we need to describe a measurement? We must know what unit was used. In the case of the fish tank, we compared the volume of the tank with the volume of a quart bottle. We call a quart the unit of volume. Thus we describe the measurement, 15 quarts, completely by giving the measure, 15, and the unit, quart.

In general, a measuring unit is some object or event which we pick for purposes of comparison. If we are told that the area of a house lot is $2\frac{1}{2}$ acres, we mean the measurement is $2\frac{1}{2}$ acres. The unit "acre" reminds us that the measurement was performed by comparing the house lot to a piece of land whose area is 1 acre. Whenever measurements result in very large or very small numbers, it is usually more convenient to use units which are more nearly of the same size as the quantity which we wish to measure. For example, in measuring the size of a building, we would use feet rather than miles, but in measuring the distance across the country, we would rather use miles.

To sum up, we can say that scientists are concerned with measurements. Measurement is the process of comparing an object or event with some unit which we have chosen. The comparison process needs to be carefully described; it always gives a number, known as a measure. This number, or measure, will depend on the choice of unit.

Exercise 1

1. Pick out the measures and the units in each of the following measurements. What might each measure?
   (a) 8 acres
   (b) 760 yards
   (c) 27 pounds per square inch
   (d) 11 fathoms

2. If a bath tub were filled by emptying a gallon bucket into it 30 times, what would be the volume of the bath tub? What is the measure? What is the unit?

3. The bath tub of Problem 2 is filled by using a quart container rather than a gallon bucket.
   (a) Does the volume remain the same?
   (b) Is the measurement the same?
   (c) What is the measure?
   (d) What is the unit?

4. Change each of the following measurements to an equal measurement having a different measure and unit.
   (a) 3 minutes
   (b) 2 pounds
   (c) 4 yards
   (d) 9 square feet

1.3 The Process of Measurement

Let us talk about the measurement of length. To give meaning to the idea of length we will suppose that length means the same thing as the distance between two points. We will often use either term to mean the same thing.

Suppose that someone asked you for the distance from Alaska to Texas. This type of question is a very common one, but is it really fair? It does

not even make sense, does it? What distance? Between which points? We need to know more about what the question really means before we can answer it.

On the other hand, people talk about the distance from Alaska to Texas as though they knew exactly what they were talking about, and they can't all be wrong all the time. This apparent difficulty can be settled as follows: It is quite true that there are many different distances between points in Alaska and points in Texas, but for many purposes, all we want is some general idea of the average distance between all these various points. When we come to consider some "ideal" distance (such as an average distance) we can say that we are making a mathematical model of the physical situation. In this model a definite meaning is given to the phrase "the distance from Alaska to Texas".

When we make a mathematical model we think about a "perfect" object or event. This way we can give a definite meaning to dimensions. We can make a mathematical model of the distance from Alaska to Texas by choosing a single point near the center of each. We could select the geographical centers or the population centers depending on what we wished to learn. The distance between the points we select could be called the distance from Alaska to Texas.

As another example let us consider the problem of measuring the width of your desk. Can you say exactly where your desk starts and where it finishes? Isn't it rather ragged and battered over the edges? Isn't it slightly wider in some places than in other places?

It is beginning to look as though the width of your desk is not, after all, that simple measurement that we hoped it was. But we must rescue ourselves quickly from the difficulties we are creating, and we do this by making a mathematical model of the desk. We now imagine that the desk is perfectly rectangular, beautifully smooth with absolutely sharp edges; and as soon as we imagine this, then we are past our immediate difficulties. There is no doubt that this ideal model of the desk has a definite width. How could we find the width of this mathematical model? The answer to this will vary from desk to desk, and from person to person, and all we can do is to give you some possibilities. You could, for example, make two very small marks, much smaller than you could actually measure, one at each end of the desk, and you could say that the distance between these two marks is THE width of the desk. Alternatively, you could lay two sharp-edged planks along the edges of the desk, so that the planks stick up slightly above the desk top. You could then measure the shortest distance between the inside faces of the planks. You should be able to think of other ways yourself of making a mathematical

model of your desk and of indicating the width of the desk.

In the rest of this book, we are going to assume that in any physical situation, all the measurements we talk about are based on some mathematical model.

Exercise 2

1. How would you find the length of your school building?
2. What are several ways of timing a 50-yard race?
3. How could you weigh yourself if no scales were available?
4. How could you compare the areas of two table tops if you had no ruler?

1.4 Measurement of Length: General

You may think it strange to start learning about measurement in general by concentrating on length. But the truth is that length is one of the most fundamental of all physical properties. The idea of length constantly comes up in both science and mathematics.

To fix our thoughts, let us talk about measuring the width of our desk. We have decided upon a mathematical model of the desk so that we can identify the width of the desk with the length of the line segment, or the distance, between two tiny marks which we have made on the two sides of the desk, for example. The problem now is how to state the distance between these two marks.

"Easy!" you will say: "It's 2 feet 4 inches," (or whatever it may be). But then we can ask: do you mean that the length is exactly 2 feet 4 inches? Or do you mean that it is somewhere near 2 feet 4 inches?

It is unlikely that you would claim that it is exactly 2 feet 4 inches, for the simple reason that you could not prove it. (Remember that "exact" means exact! Not even a millionth of an inch is allowed for uncertainty either way.)

So what you mean is that it is nearly 2 feet 4 inches. But how near? Within an inch either way, or a foot, or a quarter of an inch? It is only when questions like this are considered seriously that it becomes necessary to be much more careful about stating what a measurement is.

1.5 Measurement of Length: Ideas of Accuracy

The problem of measuring a length cannot be separated from the purpose of the measurement and the use to which it is put. Let us give some examples.

The distance of the sun from the earth is usually given as 93,000,000 miles. But, obviously, even our mathematical model of the distance is unlikely to be precisely ninety-three million miles; it is much more likely to be some figure like 93,271,412 miles! Frequently we can neglect the 271,412 miles because it is such a small fraction of the entire distance. But it would be absurd to say that we will not bother about 271,412 miles if we were going to drive from New York to San Francisco. For this distance, we must measure more accurately and might give it as 3050 miles. Our mathematical model of the distance from one particular point of the Fairmont Hotel, S.F., to one particular point of the Empire State Building, N.Y., might be more exactly 3053 miles 426 yards. So when considering driving this distance we do not bother about the 3 miles 426 yards.

And so we might argue further by showing that "the 3 miles 426 yards" would matter a very great deal if we want to measure the length of the big field nearby. Measure your height. You would not give this measurement in yards; you would want to know feet and inches.

All this can be summarized as follows: whenever a length measurement is made, it is made for some definite purpose, and this purpose will usually suggest the most suitable unit of measurement.

The unit is related to the biggest length one is prepared to neglect. Examine the following table and you will see what we mean by unit:

Measurement	A possible unit
Distance from Sun to Earth	A million miles
Distance from S.F. to N.Y.	10 miles
Length of a field	1 yard
Length of a table	1 inch
Your height	$\frac{1}{4}$ inch

This table indicates that we may be satisfied if we know the number of yards in the length of the field. For most purposes we do not want to know the number of inches from the sun to the earth, or the number of yards from San Francisco to New York. Our choice of a unit for any measurement is related to the accuracy we require or the error we are willing to accept. When we

say that the unit for the length of the field is 1 yard, we mean that we are prepared to take 73 yards as a description of its length even though the accurate length may be 73 yards 1 foot. Even though the error in taking 73 yards as the length is 1 foot, we are prepared to ignore it. Thus another way of arriving at a unit for a measurement is to consider what the maximum acceptable error is.

Exercise 3

1. Suggest suitable units for the following measurements.
   (a) the altitude of an airplane;
   (b) the length of a car;
   (c) the depth of the ocean;
   (d) the width of a window frame;
   (e) the width of a door frame;
   (f) the height of a truck.
2. What unit of measure would be acceptable when measuring the width of a window for drapery rods?
3. What unit of measure would be acceptable when measuring the width of a window glass?
4. What statement concerning choice of units of measurement is demonstrated by your answers to the questions above?

1.6 Addition of Lengths

In the previous section we saw that we can often use a rather inexact estimate of a length without knowing its precise measurement. In fact, all practical measuring processes only give estimates of a measurement. Some estimates are better than others, but we can never make an exact measurement. To know how an estimate and a measurement are related we need to understand two new ideas, namely, the addition of lengths and the difference between lengths. In this section we will discuss addition of lengths.

First of all, we must point out that although you already know a lot about the addition of numbers, the addition of lengths is a new idea and cannot be understood from mathematical ideas alone. You will later come to the ideas of addition of volumes, of masses, of times, each of which must be explained somewhat differently.

To be definite, think of two straight sticks as shown in Figure 1, one having length U and the second having length V.

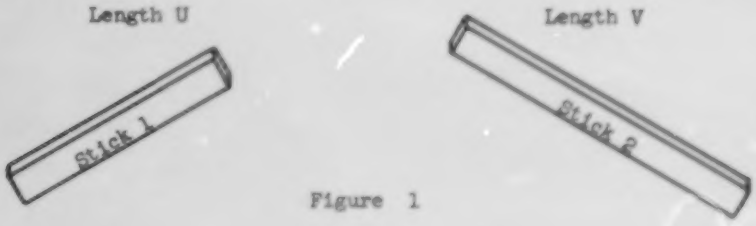


Figure 1

Since neither U nor V are numbers, (they can't be -- measurements are never just numbers) the symbol U + V does not have a definite meaning yet. We define the sum U + V to be the length of a new stick manufactured from stick 1 and stick 2 by laying them end-to-end in a straight line with no overlap, as shown below in Figure 2.

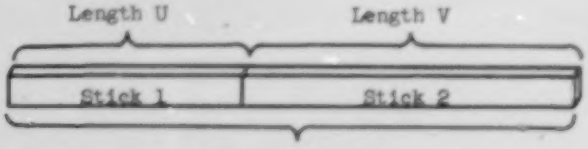


Figure 2

Notice that U and V here are single symbols which are used to represent the complete measurements. In practice, U and V would be given as numbers together with units. That is, we might have

U = 17 inches
V = 24 inches

In this case we would have

U + V = 41 inches.

The result is obtained by adding the two numbers together and using the unit common to both measurements. Notice that this can be done only when both measurements are given in the same units.

The sum of the two lengths is defined as above no matter what units might be used in expressing the measurements. That is, if

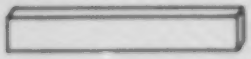
U = 17 inches
V = 2 feet

then we still have

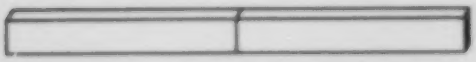
$$U + V = 41 \text{ inches.}$$

In later sections we will say more about the actual computations. In this section we are more interested in the physical idea. The sum of two lengths $U + V$ is the length of a new stick as shown in Figure 2.

Suppose that both of the lengths U and V are the same. Then the sum $U + U$ would be the length of the combination of two identical sticks, each of length U, as shown in Figure 3.



Length U



Length $U + U$

Figure 3

A good name, and somewhat shorter, for $U + U$ would be 2U. If U is a symbol standing for a measurement such as 17 inches, then 2U is a symbol standing for a measurement twice as long, 34 inches in this case. However, we can look at the symbol 2U in another way. We can think of 2U as meaning the measurement whose measure is 2 and whose unit is a stick of length U. In this case, the symbol U behaves like a unit. Although U is actually a measurement, we can use it as a unit without difficulty. From this point of view, when we write 2U, we mean the measurement whose U measure is 2.

If n is any counting number, we can now give meaning to the symbol nU. Namely,

$$nU = \underbrace{U + U + U + \ldots + U}_{n \text{ terms}}$$

Then, nU is the length of the stick formed by combining n sticks (in the proper way, of course, as shown in Figure 4 each with length U.

Length U

Length nU

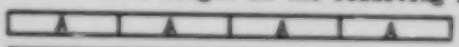
n sticks

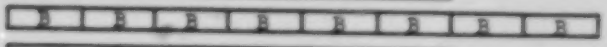
Figure 4

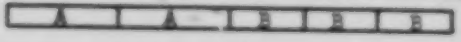
The longer stick in the diagram evidently has a length of nU. The U measure of the length is n. You should keep clearly in mind that the symbol nU only has meaning (so far) when U is a length and n is a counting number (1, 2, 3, ...). Symbols such as $\frac{2}{3}$U or 0.123U are explained later, since they are more difficult.

Exercise 4

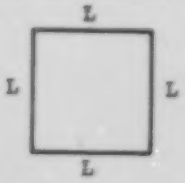
1. What is the total length in the following figures?

(a) 

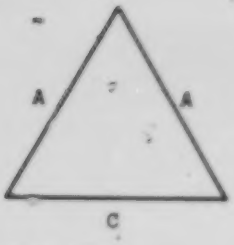
(b) 

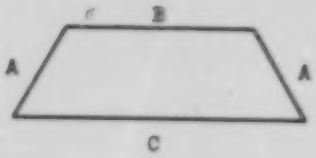
(c) 

2. What is the perimeter of each of the following figures?

(a) 

(b)

(c) 

(d) 

1.7 Unequal Numbers

We have seen that a given measurement may be described in different ways, which depend on the units chosen. In Section 1.2 we stated that 15 quarts and 30 pints are both names for the same measurement. Therefore,

15 quarts = 30 pints.

Here we use the "=" sign exactly as it is used in mathematics, to declare that the name on the left and the name on the right are names for the same thing.

Now let us describe the possible relationships between lengths that are not equal. 3 yards and 8 feet, you will surely agree, are not equal lengths. Is there a simple way of describing how they are related?

Recalling some properties of numbers may give us an idea.

When two numbers are compared as to size, there are three possible results. The numbers may be equal, or the first may be greater than the second, or the first may be less than the second. These relations between the numbers can be expressed very conveniently by the use of the following symbols:

Symbol	Read
" = "	is equal to
" > "	is greater than
" < "	is less than

for example

$3 = 3$	is read as "three is equal to three"
$16 > 3$	is read as "sixteen is greater than three"
$3 < 10$	is read as "three is less than ten".

Notice that the point of the "<" or ">" symbol is always nearer the smaller number and the open end is nearer the larger number.

The two symbols ">", "<" are called inequality symbols. Notice that

$$16 > 3$$

says exactly the same thing as

$$3 < 16$$

so that if we don't like one form, we can use the other.

Exercise 2

1. Use one of the symbols, "<", "=", ">", to make a true statement for each of the following pairs of numbers.

(a) 5,6
(b) 5,10
(c) 16,16
(d) 3,8
(e) 19,11
(f) 2,7
(g) 9,4
(h) $\frac{1}{4}$,.25
(i) $\frac{13}{15}$, $\frac{3}{4}$
(j) 7,a

1.8 More on Inequalities

We say that $2 < 7$ and $5 < 6$ are inequalities in the same sense; so are $4 > 1$ and $16 > 10$. But $1 < 3$ and $4 > 2$ are inequalities in the opposite sense. So are $26 > 16$ and $11 < 14$. "Same sense" suggests that both symbols point in the same direction; "opposite sense" suggests they point in opposite

directions.

Two inequalities in the same sense may "overlap" if the larger number of one is the smaller number of the other. For instance, $3 < 10$ and $10 < 16$ are overlapping inequalities in the same sense. The same information can be written more briefly as $3 < 10 < 16$, which is read "3 is less than 10 and 10 is less than 16". In general, for three numbers a, b and c, the statement

$$a < b < c$$

means

$$a < b \text{ and } b < c.$$

> In mathematics, a two-part statement using the conjunction "and" is true only when both parts of the statement are true.

The statement $a > b > c$ means $a > b$ and $b > c$. Thus $9 > 4 > 1$ is a true statement because $9 > 4$ and $4 > 1$ are both true.

Notice that we must have inequalities in the same sense before we can combine them in this way. Mathematicians do not use such symbols as $a < b > c$ or $a > b < c$. In $a < b > c$, we know that b is greater than a and b is greater than c, but we do not know how a and c compare. Can you state in words $a > b < c$?

If a and b are two numbers about which we know only that a is not greater than b, we can say with confidence that at least one of the two remaining possibilities is true. That is, either $a < b$ or $a = b$.

> In mathematics, a two-part statement using the conjunction "or" is true if either part or both parts are true.

It is useful to be able to combine the two parts of the statement "$a < b$ or $a = b$" into a single symbolic statement: $a \leq b$. The statement $a \leq b$ means $a < b$ or $a = b$. The line under the inequality symbol is supposed to suggest an equality symbol to us. We read "$a \leq b$" as "a is less than or equal to b". $2 \leq 3$ is true because the second of the statements $2 = 3$, $2 < 3$ is true.

Similarly, $5 \leq 5$ is true since the first of the statements $5 = 5$, $5 < 5$ is true.

The statement $10 \leq 4$ is false, because neither of the statements $10 = 4$ or $10 < 4$ is true.

To summarize, if a and b are numbers,

"$a \leq b$" means "either $a = b$ or $a < b$".

Now we can invent a most helpful form of inequality that will be used over and over in describing how accurate an estimate of a length may be.

If a, b, and c are numbers, and

$$a \leq b < c,$$

we mean that $a \leq b$ and $b < c$.

Thus $5 \leq 7 < 9$ is true because $5 \leq 7$ and $7 < 9$.
$5 \leq 5 < 9$ is true because $5 \leq 5$ and $5 < 9$.
$5 \leq 9 < 9$ is false because $9 < 9$ is false.
$5 \leq 4 < 9$ is false because $5 \leq 4$ is false.
$5 \leq 10 < 9$ is false because $10 < 9$ is false.

Exercise 6

1. Rewrite each pair of inequalities below so that they are in the same sense. Write them as overlapping inequalities whenever possible.

   (a) $3 < 4,\ 5 > 3$
   (b) $7 > 5,\ 7 < 13$
   (c) $16 < 21,\ 21 > 19$
   (d) $a \leq b,\ c > b$
   (e) $31 > 25,\ 21 < 30$
   (f) $a < c,\ d > a$
   (g) $m > n,\ m < 1$
   (h) $p < q,\ q > t$
   (i) $16 > 7,\ 4 \leq 7$
   (j) $d > e,\ f < c$

2. What can be said of two numbers, a and b, if we know that $a \leq b$ and also $a \geq b$? Explain your reasoning carefully.

3. Complete the following table by indicating in the proper space whether each part is true or false and whether the compound statement is true or false. (Review the boxed definitions of the conjunctions "and" and "or" as used in mathematics.)

Statement A	Conjunction	Statement B	A	B	compound statement
Example: $5 < 3$	and	$4 < 5$	F	T	F
(a) $5 < 3$	or	$4 < 5$			
(b) $17 > 32$	and	$7 > 6$			
(c) $(5-1) > (2+2)$	or	$(\frac{10}{2}) > (6-1)$			
(d) $\frac{5}{7} < .72$	or	$.72 < \frac{5}{7}$			
(e) $17 > 9$	and	$1\frac{2}{6} > 1.8$			

4. Complete the following table by separating the compound statements into two parts. Tell whether each part is true or false, determine the conjunction which is indicated, and tell whether the compound statement is true or false.

	Statement A	Conj.	Statement B	A	B	compound statement
Example: $3 < 4 < 5$	$3 < 4$	and	$4 < 5$	T	T	T
(a) $5 \geq 4$						
(b) $3 > 2 > 4$						
(c) $6 \leq 6$						
(d) $2\frac{3}{8} \geq 2.375$						
(e) $\frac{13}{22} > \frac{30}{51} > \frac{40}{69}$						

5. Answer true or false for each of the following.

(a) $5 \leq 5 \leq 6$

(b) $5 < 5 \leq 6$

(c) $5 \leq 5 < 6$

(d) $5 < 5 < 6$

1.9 Unequal Lengths

So far we have discussed only unequal numbers. Nothing has been said about unequal lengths. When can we say that one length is greater than another length? We must caution you at the outset that length inequality is not the same idea as number inequality and that what you know about number inequality by itself can only suggest what is true of unequal lengths.

We agreed that a length of 3 yards and a length of 8 feet were not equal. Does that mean that 8 feet is a greater length than 3 yards because $8 > 3$? Clearly not. Measure inequality (that is, number inequality) by itself tells us nothing about length inequality.

We need a different approach.

If a length of 1 foot is added to 8 feet we get 3 yards. "8 feet + 1 foot" is the length of a stick constructed from two sticks, one 8 feet long and the other 1 foot long. The length of the combination is 9 feet, a measurement which we recognize as the same as 3 yards.

It is just as clear that there is no length that can be added to 3 yards that will give 8 feet.

Therefore we are inclined to say that 3 yards is greater than 8 feet rather than the other way around.

In general, a length U is greater than a length V when $U = V + T$ for

some length T. The symbolic statement U > V is read "U is greater than V". When there may be doubt about whether length inequality or number inequality is meant, we could be cautious and write (length)U > (length)V to remind ourselves that length inequality is intended. Generally the statement U > V will be sufficient.

Let us sum up all the possibilities. Whenever U and V are any two lengths one, and only one, of the following statements is true:

$$U = V, \quad U > V, \quad \text{or } V > U.$$

To say V < U merely means U > V, just as with numbers.

Exercise 7

1. For lengths U, V, and W, write in words the following statements.
   (a) $U < V < W$
   (b) $U \leq V$
   (c) $U \leq V < W$

2. Determine which statements below are true and which are false. Give reasons in each case.
   (a) 3 feet $\geq$ 39 inches
   (b) 42 inches $\leq 3\frac{1}{2}$ feet $< 1\frac{1}{3}$ yards
   (c) 40 inches > 2 feet > 1 yard

1.10 Measurement: A Classroom Experiment

Each member of your class should bring a stick to school to use as the unit for measuring the length, or width, of the classroom. You could pretend that you are measuring it for a carpet, so that you will then have an idea of what standard of measurement is suitable. This will determine, roughly, the length of the stick you will bring.

Question 1. Will your stick be about (a) $\frac{1}{4}$ inch, (b) 1 foot, or (c) 2 yards long?

During the lesson each one of you should measure the length of the room, using your stick as your chosen unit of measure. As you will have only one stick unit and not a whole lot of exactly equal ones, you will mark the lengths off one by one, using a pencil or light chalk mark on the floor or wall to indicate where the stick ends before shifting it. You will find that a certain largest number of stick-lengths can be placed end-to-end across the room.

Suppose, for example, that 34 stick lengths can be placed across the room, but that there isn't room for a 35th stick. Then you would know that the measure, d, of the room (in stick units) could be stated by the inequality

$$34 \leq d < 35.$$

Each one of you, therefore, will arrive at an inequality. We suggest that your teacher then make a table, like this one, on the blackboard.

Member of class	Inequality for measure
Anne	$34 \leq d < 35$
Bob	$12 \leq d < 13$
Carla	$30 \leq d < 31$
Don	$29 \leq d < 31$
Ethel	$39 \leq d < 40$
Frank	$62 \leq d < 63$
and so on	

You will be able to discuss the results you get in a number of ways. The following questions (which refer to our table printed above) will give you some clues as to what conclusions can be drawn:

Question 2: Who has failed to perform the measurement according to the instructions?

Question 3: Who used (a) the largest stick,
(b) the smallest stick?

Question 4: Whose sticks were closest to each other in length?

Question 5: Would you choose one of these measurements as the best, or the most convenient, or the most reliable measure? If so, which? And why?

Question 6: In what circumstances would all the inequalities be the same?

Question 7: If everyone had used a brand-new piece of chalk for his or her measure unit, should all the inequalities have come out the same?

Question 8: If pieces of chalk were the only measuring unit available, would it be possible to go and buy the right size of carpet in the local market?

Question 9: What advantages, if any, would arise from using sticks exactly 1 foot long rather than pieces of chalk?

1.11 Standard Units

You will have realised from the preceding experiment for measuring the classroom length that it would be extremely awkward in practice if each individual used his own individual measure stick. Many of the squabbles between primitive peoples in olden times must have been due to different people using different measure sticks for the length of, say, a field: the man who was selling the field would say it was 55 paces long and the man buying it would say it was 49 paces long. Both might be right because each was using the length of his pace as his unit and these were of different lengths.

Thus, the practical necessity, and it is no more than practical, arises for units of measure which are common to everyone. The standard units of length (as they are called) are the meter and the foot. These units are the ones agreed upon by the leading governments of the world. Can you see why there is need for standard units of measure in order to share the scientific knowledge of the world?

The foot is the traditional British standard unit of length which is widely used in English-speaking countries. You will probably already know how it is related to other common British measures: inch, yard and mile.

The meter is the unit in everyday use in the Continental European countries, and is also very widely used by most scientists all over the world. Among other measures related to the meter are: millimeter, centimeter and kilometer.

We shall discuss these various measures in the next chapter.

1.12 Length Measurement

Let us consider an object whose length is represented by the letter L. To find a measure of the length L we must first choose a suitable unit of length for comparison; let us call the unit U. Then the measure of L in terms of the unit U will be some number d. We now describe a counting procedure that will give us an estimate of d.

We shall assume that we have available a large number of sticks, each with length U. Next, as is illustrated in Figure 5, the sticks are arranged end-to-end along the line segment which defines the length of the object, starting at one end.

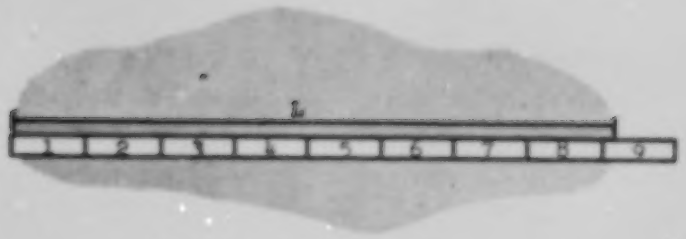


Figure 5. Measurement with unit sticks

We count the greatest number of sticks which can be arranged as in Figure 5 without ever overlapping the other end and we call this number n. In Figure 5, n = 8. Now, it seems plausible that we must choose d between the numbers n and n+1, that is, we must have

$$n \leq d < n+1. \tag{1}$$

The sign $\leq$ indicates that d could actually be equal to n if the end of the object coincided with the end of a stick.

Let us use the symbol nU to represent the length of n sticks placed end-to-end. Thus in the figure the length of the object is larger than 8U and shorter than 9U. In general, we can write

$$nU \leq L < (n+1)U. \tag{1A}$$

If this inequality is compared with (1), it seems reasonable to require that the measure d should be so chosen that L = dU, because then we shall have

$$nU \leq dU < (n+1)U.$$

Notice that the length measure d has <u>not</u> been exactly determined. What is more, because there is no practical way of making absolutely certain that the ends of two objects are precisely lined up, no measurement can be made exactly. There will always be an uncertainty, and this uncertainty may lead to an inequality such as that in (1).

1.13 A Property of Order: The Transitive Property

In Section 1.8 we explained what was meant by inequalities "in the same sense", both for numbers and for lengths. We also pointed out that if two such inequalities "overlap" (the larger element of one inequality being the smaller element of the other), they can be expressed as an overlapping inequality. For example, if $6 < 7$ and $7 < 30$, we can write $6 < 7 < 30$.

Now we ask, what can be said about the numbers that are not mentioned twice, such as 6 and 30? Can we tell that $6 < 30$? Certainly. How? By direct comparison.

Let's try again. What can we tell about numbers a and c if all we know about them is that $a < b$ and $b < c$ for some number b? This is a harder question, because we cannot compare a and c directly. But our long experience with numbers still makes us believe that $a < c$.

Let us state our belief again, briefly.

For numbers a, b and c, if $a < b$ and $b < c$, then $a < c$.

This is in fact a property of our number system.

This property of numbers is so important that we will find ourselves referring to it again and again. It will be convenient to give it a special name. The statement in the above box is called the transitive property of number inequality. We can state the transitive property in words.

If the first of three numbers is less than the second, and the second is less than the third, then the first is less than the third.

Now you may not believe the statement in the box. If so, you should pause a bit to try to think up three numbers such that the first is less than the second and the second is less than the third, but the first is greater than or equal to the third. Like all the properties of our number system, the transitive property can be used over and over. For instance, if $a < c$ and $c < d$ and $d < b$, can we say anything about a and b? Certainly. Since $a < c$ and $c < d$, we know that $a < d$. This new fact, $a < d$, together with $d < b$, tells us that $a < b$, if we use the transitive property once again.

Measurements, we have insisted all along, are not numbers. Do you suppose that there is also a transitive property of length inequality? Do you believe that if the first of any three lengths is less than the second and the second is less than the third, then the first must be less than the third? Or, to put it more briefly in symbols, do you believe the following statement?

For lengths U, V, and W,
if $U < V$ and $V < W$, then $U < W$.

If you are not yet ready to believe it, there are two courses open to you. First, you could try to find three lengths for which the transitive property fails. If after many trials you are unable to find three such lengths, you may begin to suspect that your task is impossible, or, to put it another way, that length inequality <u>is</u> transitive.

A second way to convince yourself that length inequality is transitive is to ask whether it <u>must</u> be true because of the meaning we have given to the statement (length)U > (length)V. In Section 1.9 we learned that

(length)U > (length)V means there is a length T such that $U = V + T$.

That is, two lengths are unequal when some length added to the smaller gives the larger.

Now let us suppose that $U > V$ and $V > W$. To show that the inequality $U > W$ <u>must</u> be true we must find a length which added to W will give U. Then our job will be done.

Here goes.

Since $U > V$ then $U = V + T$ for some length T.
Since $V > W$ then $V = W + S$ for some length S.

Since V and $W + S$ are different names for the <u>same</u> length, we can replace V by $W + S$ whenever we wish, so

$$U = W + S + T$$

or better

$$U = W + (S + T).$$

This says that by adding a length $S + T$ to the length W we get length U. That means that W is the shorter length, or

(length)U > (length)W,

as we wished to show.

<u>Exercise 8</u>

1. State the transitive property of number inequality using "greater than".
2. Is there a transitive property for equality? If so, state it.

3. Test the following phrases for transitivity.

Example:

"is the father of"

If A is the father of B, and B is the father of C, it does not follow that A is the father of C. Therefore "is the father of" does not have the transitive property.

(a) "is older than"

(b) "lives within 200 miles of"

(c) "has the same mother as"

(d) "is taller than"

(e) "lives next door to"

4. a, b, c and d are numbers. State an inequality between a and d (whenever possible) in each of the following cases.

(a) $b < a$, $c < b$, $b > d$;

(b) $d > b$, $b < c$, $b > a$;

(c) $d < c$, $a > c$, $b > a$, $d < b$;

(d) $c > b$, $a > b$, $b > d$;

(e) $a < c$, $b < c$, $d > b$.

5. In the four exercises below all letters refer to numbers.

(a) If $p < q$ and $q < r$, does it follow that $r > p$? Why?

(b) If $m > p$ and $p > n$, does it follow that $m > n$? Why?

(c) If $d > f$ and $d > g$, does it follow that $f > g$? Why?

(d) If $h < k$ and $j < h$, does it follow that $k > j$? Why?

1.14 Another Property of Order: The Addition Property

Our concluding section describes a new property of inequalities in the same sense. We start off as usual with number inequalities.

We can all agree that $5 < 8$. Is it true that $5 + 1 < 8 + 1$? That $5 + 11 < 8 + 11$? That $5 + 126 < 8 + 126$? Checking each inequality separately we see that $6 < 9$, $16 < 19$, and $131 < 134$ are all true statements so the answer is "Yes" in each case. Does it seem to matter which number we choose to add to both sides?

Let us try to sum up what we are beginning to suspect. Let us use a, b and c for the names of numbers.

If $a < b$, then $a + c < b + c$.

This is indeed a property of our number system, the addition property of number inequalities.

> If equal numbers are added to both numbers of an inequality the sums are unequal in the same sense.

Knowing this you should be able to prove the following statement.

$$\boxed{\text{If } a \leq b, \text{ then } a + c \leq b + c.}$$

Now for something a little more ambitious. We start with two number inequalities in the same sense: $7 > 4$ and $9 > 1$. Is the sum of the two larger numbers greater than the sum of the two smaller numbers? Is it true that $7 + 9 > 4 + 1$? Since "$16 > 5$" is a true statement, we can say yes.

Do you think that it will always work out like this? More precisely, if all you know about four numbers a, b, c and d is that $a > b$ and $c > d$, would you feel very sure that

$$a + c > b + d?$$

Here is a good way to answer this question. We use what we know about adding equal numbers to both numbers of an inequality.

Since $a > b$, $a + c > b + c$ — c added to both numbers

Since $c > d$, $b + c > b + d$ — b added to both numbers

In the box we see two overlapping inequalities in the same sense. Therefore we can use the transitive property of number inequality to get

$$a + c > b + d.$$

Let's sum up our findings.

$$\boxed{\begin{array}{l}\text{For any four numbers a, b, c and d,}\\ \text{if } a > b \text{ and } c > d, \text{ then}\\ \qquad a + c > b + d.\end{array}}$$

The statement in the box can be summed up in words as follows:

The sum of two inequalities having the same sense is a new inequality having the same sense.

It is particularly easy to use this addition property when the inequalities are arranged one on top of the other as shown below:

$$\begin{array}{r} a > b \\ \underline{c > d} \\ a + c > b + d \end{array}$$

Here the two given inequalities appear above the line, and the resulting inequality appears below the line. The sums below the line are formed from the numbers directly above them.

Arranging our work in this way makes it easier to work with a pair of overlapping inequalities in the same sense. Suppose that a and b are numbers for which $5 > a > 3$ and $13 > b > 7$. What can we say about the sum of a and b? How big can it be? How small can it be? The answer comes quickly when we rearrange the inequalities as follows:

$$\begin{array}{r} 5 > a > 3 \\ 13 > b > 7 \\ \hline 5 + 13 > a + b > 3 + 7 \end{array}$$

or $$18 > a + b > 10$$

This shows us that the sum of a and b must be less than 18 and greater than 10.

So far in this section we have spoken only of number inequalities. Similar statements can be made about length inequalities. Each of them can be proven true by using the meaning of the statement (length)U > (length)V. We list them here for your use without proving them.

Suppose U, V, S and T are any lengths.

If $U > V$, then $U + S > V + S$.
If $U > V$ and $S > T$, then $U + S > V + T$.

Exercise 9

1. Given the four numbers, R, S, T, U where $R > S$ and $T > U$. Prove that
$$R + T > S + U.$$
Suggestion: add T to both numbers of $R > S$ and add S to both numbers of $T > U$. Now apply the transitive property to the two new statements.

2. What can be said about the sum of the numbers x and y in the following?
   (a) $2.2 < x < 2.5$ and $4.1 < y < 4.5$
   (b) $3 > y > 1$ and $16 < x < 18$
   (c) $11 < y$ and $x > 9$
   (d) $7 < y$ and $x < 7$

3. Suppose that a, b, c and d are numbers.

(a) If $a < b$ and $c \leq d$, how do the sums $a + c$ and $b + d$ compare?

(b) If $a \leq b$ and $c \leq d$, how do the sums $a + c$ and $b + d$ compare?

1.15 Summary

Measurement is the process of comparing some object or event with some unit which we have chosen. This comparison process always gives a number, known as a measure. This number, or measure, will depend on the choice of unit. In making measurements we make a mathematical model of some physical situation. When we make a mathematical model we think about a "perfect" object or event. This way we can give a definite meaning to dimensions.

Whenever a length measurement is made it is made for some definite purpose. This purpose will usually suggest the unit of measurement we should choose. Our choice of a unit for any measurement depends on the accuracy we require or the error we are willing to ignore. The needs of people to share ideas have resulted in the selection of standard units of measure.

When numbers are compared as to size, there are three possible results. The numbers may be equal, the first may be greater than the second, or the first may be less than the second. The sense of a statement of inequality indicates whether the first number is greater than the second or the second number is greater than the first.

Two important properties of order are the transitive property and the addition property. The transitive property shows how we can compare the extreme ends of an overlapping inequality. The addition property lets us add the same number or measurement to both sides of an inequality.

Chapter 2

LENGTH AND THE NUMBER LINE

2.1 Using Related Units in Measuring

Let us consider a definite problem, namely, the measurement of the length of our desk. Remember that first we have to make a mathematical model of the desk because in fact it is chipped and rough and crooked. Having made this model, we now have two definite points to measure between.

Next we have to choose a unit of measure. Normally we would pick one of the standard units. However, we may wish to compare the length of this desk, as accurately as we can, with that of some other desk of the same type. Are the desks the same length or do they differ in length? To get a good comparison we choose a quite small unit of measurement such as the millimeter. This is a small unit of length about as long as the distance between these two lines || .

Suppose your desk is roughly 1200 millimeters long. Let us assume, for the sake of the argument, that your measure of length is 1253 millimeters. Remember that this result could have been obtained just by counting millimeter sticks laid end to end; one thousand, two hundred fifty three of them! In practice, what we would have done is illustrated in Figure 3 of Chapter 1.

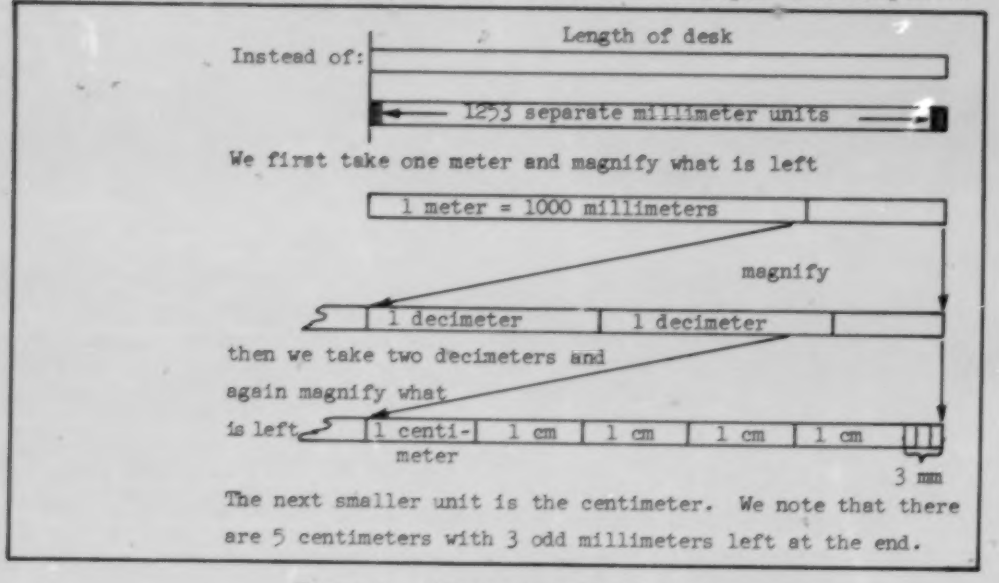


Figure 1. Measurement of a Desk

Finally, we can write for the millimeter measure of the desk, d,

$$1253 \leq d < 1254.$$

We have actually obtained the numbers 1253 and 1254 by taking batches of decimal groupings. The 1253 has really been constructed as

$$1(1000) + 2(100) + 5(10) + 3(1).$$

The measure numbers 1000, 100 and 10 which occur in this statement are called compound measures: they are not the basic measure but are certain carefully chosen multiples of the basic measure. In fact, each is ten times the succeeding one. For this reason, such a system of counting is called the DECIMAL SYSTEM.

Measurement, as we have seen, involves considerations over and above those of counting. However, throughout the measuring problem, it is more convenient to use a system which is based on the decimal system in order to simplify the counting that must be done. This is provided by the METRIC SYSTEM of measures.

2.2 The Unmarked Stick Experiment

To help us understand the ideas of measurement and the need for subdivision of the unit of measurement, we shall perform an experiment.

Bring an unmarked straight stick about 15 inches long to school. Suppose you use this stick as your unit of length U. With this stick, measure the height of the door of the classroom, the length of one of your books and the width of a room. On a sheet of paper in your notebook make a table similar to the one in Figure 2. Record the results of the measures you have made in the form of inequalities. Your table might look like this, but the numbers you assign as measures will probably be different.

Table I: Measurement in Units
$4 \leq$ height of door < 5, nearer to 5
$0 \leq$ length of book < 1, nearer to 1
$15 \leq$ width of room < 16, nearer to 15

Figure 2

The estimates given in your table are probably not good enough for most

purposes. To make better estimates of these lengths, you may consider dividing the stick into equal parts, for example, into tenths of a unit so that you could make statements such as

$$4.8\ U \leq \text{height of door} < 4.9\ U.$$

Let us see how we could divide our stick into ten equal parts. Take two sheets of ruled notebook paper. Check that the lines on these sheets are equally spaced by moving one sheet along the other as illustrated in Figure 3.

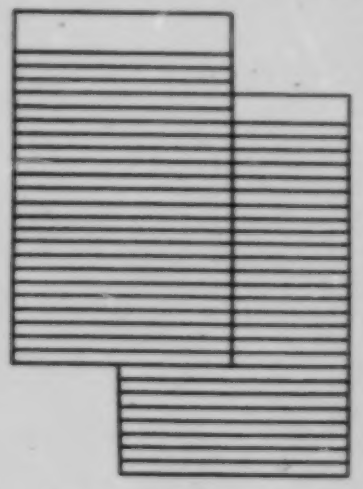


Figure 3. Test for Equally-Spaced Lines

Tape two sheets together so that some of the lines on the top of the second sheet are aligned with some of the lines on the bottom of the first sheet. See Figure 4.

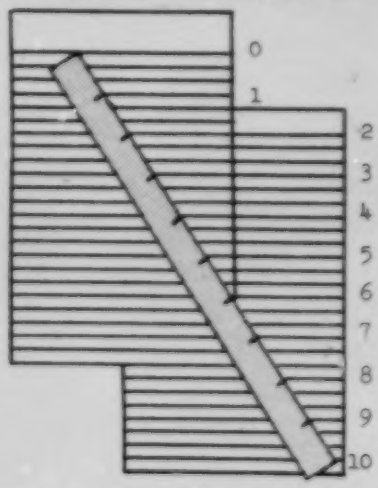


Figure 4. Subdividing the Stick

Number the topmost line 0, skip the next two lines and number the third line 1. Skip the two following lines and number the next line 2 and so on

until you reach number 10. Place one end of the stick as nearly as possible on the line which was numbered 0, and adjust the stick diagonally across the sheets of paper as in Figure 4, so that the other end lines up with the line marked 10. If the stick overlaps both sheets, you may have to shift your second sheet down and retape. Mark, with the numbers 1 to 9 inclusive, the points where your numbered lines meet the stick.

Use this marked stick to measure the same lengths as in Table I. This time you will allow the stick to extend beyond the edge of the desk on the last act of measuring and count the number of divisions of the stick which do not extend beyond the edge of the desk. (See Figure 5.) The entries should be on the form: $4.8 \leq$ height of door < 4.9, nearest 4.9, and so on. Record your data and label it "Table II".

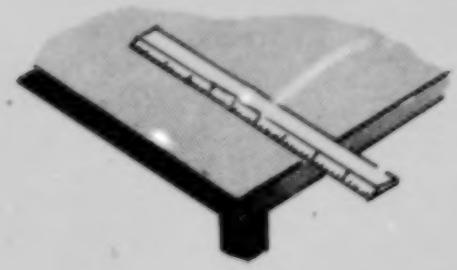


Figure 5. Measuring with Marked Stick

To find a still more accurate estimate of the lengths we are measuring, we must use a finer division of the stick. Repeat the method originally used to divide the stick, but use paper with lines much closer together. Choose a piece of graph paper such as that shown in Figure 6. Number a horizontal line on it as the 0 line, and number the successive horizontal lines from 1 to 10 inclusive.

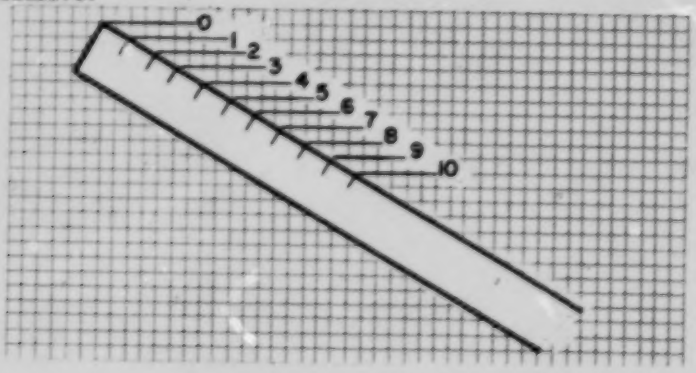


Figure 6. Dividing into Hundredths

Place one end of the stick on the line numbered 0 and adjust the stick so that the line numbered 10 on the graph meets the stick at the first mark (numbered 1) you made before. Now mark (do not number) the stick at each point where the stick crosses a numbered line on your graph. To divide the next interval on your stick, slide the stick up the paper until the first mark numbered 1 is on the line of the paper numbered 0. Adjust the stick so that the mark numbered 2 is on the line numbered 10 on your graph paper. Again, mark the stick at each point where the stick crosses a numbered line of your graph. Repeat this procedure for each of the original divisions on your stick. It will now be divided into one hundred equal intervals of length.

Repeat the experiment of measuring the door, the book, the room, and record your data in Table III in the same form as before.

Exercise 1

1. Explain why moving the sheets along each other as indicated in Figure 3 shows that the lines are equally spaced.

2. How do you know that your procedure for dividing the stick into ten parts of equal length really works? Devise a method to divide the stick into seven equal parts.

3. If you are measuring your desk with a meter stick marked in divisions down to millimeters, how many decimal places will you be able to give in the measure inequality if the measure is based on units of
   (a) Meters?
   (b) Centimeters?
   (c) Kilometers?

4. Measure the length of your neighbor's stick with your stick, and find its measure to the nearest hundredth of your unit. Use this number to convert his data in Table I to measures in terms of your unit. Do the results you find this way agree with your measurements noted in Table I? If the agreement is unsatisfactory, can you give some explanation for the disagreement?

5. Convert your neighbor's data in Table III to measure in terms of your unit. Do these results agree with your measurements noted in Table III? Is the agreement between your results and your neighbor's results better or worse now than in Problem 4?

2.3 The Metric System of Length

Consider the desk of Figure 1 in Section 2.1 which we assumed was 1253 millimeters long. Its millimeter measure, d, satisfied the inequality

$$1253 \text{ millimeters} \leq d < 1254 \text{ millimeters.}$$

Another way of expressing this inequality could be

$$1253 \leq d_m < 1254$$

where the m, called a subscript, tells us that the unit of measure used here is the millimeter. The expression d_m is read "d sub m".

Suppose we were asked to find the length of the desk measured in centimeter units. We may try to answer this by using the same method used in the previous sections and counting the number of centimeter sticks which do not overlap the desk. If we call d_c the centimeter measure of the desk, we would find that $125 \leq d_c < 126$. In a similar way, if we count using decimeter sticks, we would find that the decimeter measure d_d satisfies $12 \leq d_d < 13$.

We now have three different inequalities for the measure of the same length:

$$1253 \leq d_m < 1254 \qquad (1)$$

$$125 \leq d_c < 125 \qquad (2)$$

$$12 \leq d_d < 13 \qquad (3)$$

It is obvious that these three inequalities tell us successively less and less about the length of the desk. For example, the last inequality states only that the length of the desk is between 12 and 13 decimeters, whereas the first inequality indicates a closer estimate, namely, that the length of the desk is between 1253 and 1254 millimeters.

Is there some way to show that each of these is an accurate measure for the length of the desk? Yes, since we realize that in each case we are measuring the same length.

Now we know that the metric system is a decimal system. We have already indicated that when the basic unit of one meter is divided into ten equal parts, each part is called a decimeter. In mathematics, we would say,

$$1 \text{ meter} = 10 \text{ decimeters.} \qquad (4)$$

Another way to say exactly the same thing is

$$1 \text{ decimeter} = \frac{1}{10} \text{ meter.} \qquad (5)$$

Carrying the same idea further, we can divide the decimeter into ten equal parts and say

$$1 \text{ decimeter} = 10 \text{ centimeters.} \qquad (6)$$

Similarly, we can say

$$1 \text{ centimeter} = 10 \text{ millimeters.} \qquad (7)$$

To express the length of the desk in centimeters instead of millimeters we note that

$$1 \text{ millimeter} = \frac{1}{10} \text{ centimeter} \qquad (8)$$

is another way of saying exactly the same thing as statement (7).

Since the length of the desk was 1253 millimeters, we find by using (8) that

$$\begin{aligned} 1253 \text{ millimeters} &= 1253\left(\tfrac{1}{10} \text{ centimeter}\right) \\ &= 1253 \times \tfrac{1}{10} \text{ centimeters} \\ &= 125.3 \text{ centimeters} \end{aligned}$$

In a similar way, we could find the decimeter measure and the meter measure of the desk. We begin with the fundamental relation between the units, namely, that

1 meter = 10 decimeters = 100 centimeters = 1000 millimeters.

We arrange it so that

$$1 \text{ millimeter} = \frac{1}{10} \text{ centimeter} = \frac{1}{100} \text{ decimeter} = \frac{1}{1000} \text{ meter.}$$

Then the length of the desk is

$$1253 \text{ mm} = 125.3 \text{ cm} = 12.53 \text{ dm} = 1.253 \text{ m.}$$

Notice that the four measures of the desk are related to each other. One measure can be obtained from the other by multiplying or dividing by some multiple of ten. Thus, the magnitude of these numbers differ only in the position of the decimal point in each. So 1253 millimeters (remember a decimal point is to be understood after the final digit) is equal to 125.3 centimeters, to 12.53 decimeters, and to 1.253 meters. This simple relation between the measures in terms of the different units is the reason why the metric system is so convenient to use.

This process of changing from one basic unit of measure to another without actually going through the process of measuring with the new unit is called "conversion of units". As you will notice in the examples, the expression 1253 millimeters could be thought of as 1253 millimeter units or,

more simply,

$$1253 \text{ (1 millimeter)}.$$

The conversion to centimeters can be brought about by referring to (8) and replacing the (1 millimeter) with its equivalent, ($\frac{1}{10}$ centimeter). Similarly, to convert to meters we would write

$$1253 \text{ (1 millimeter)} = 1253(\tfrac{1}{1000} \text{ meter})$$

so that we now have

$$1253 \text{ millimeter} = \tfrac{1253}{1000} \text{ meters} = 1.253 \text{ meters}.$$

Let us try two more examples of this change of units. Suppose a length is 23.7 meters. What is its measure in decimeters, in centimeters and in meters? We have

$$23.7 \text{ meters} = 23.7(10 \text{ decimeters}) = 237 \text{ decimeters};$$

also

$$23.7 \text{ meters} = 23.7(100 \text{ centimeters}) = 2370 \text{ centimeters},$$

and

$$23.7 \text{ meters} = 23.7(1000 \text{ millimeters}) = 23700 \text{ millimeters}.$$

Notice that the different measures 23.7, 237, 2370 and 23700 are obtained from each other by multiplying by some multiple of ten in order to get the correct placement of the decimal point. Similarly, to find the different measures for a length of .038 centimeter, we write

$$\begin{aligned} .038 \text{ centimeter} &= .038(10 \text{ millimeters}) \\ &= .38 \text{ millimeter}, \\ .038 \text{ centimeter} &= .038(\tfrac{1}{10} \text{ decimeter}) \\ &= .0038 \text{ decimeter}, \end{aligned}$$

and

$$\begin{aligned} .038 \text{ centimeter} &= .038(\tfrac{1}{100} \text{ meter}) \\ &= .00038 \text{ meter}. \end{aligned}$$

The measures .038, .0038 and .00038 are obtained from each other by successively dividing by ten and, in a sense, moving the decimal point one place to the left, using zeros to fill in the empty places.

Exercise 2

1. Complete the following table where each row refers to a given length:

	Millimeter Measure	Centimeter Measure	Decimeter Measure	Meter Measure
a		5672		
b			20.47	
c	.056			
d				2.3

2. Find out how the lengths, meter, dekameter, hectometer, kilometer are related, and extend the table in Problem 1 to the measures in each of the last three units.

3. Describe a counting process that could lead to a meter-measure of 23.987.

4. Fill in the following blanks: [1 kilometer = 1000 meters]

   5 kilometers = _____ meters

   256 millimeters = _____ kilometers

   1245 centimeters = _____ millimeters

   536 centimeters = _____ kilometers

5. Write another expression which says exactly the same thing as each of the following:

   (a) 1 foot = 12 inches
   (b) 1 yard = $\frac{1}{1760}$ mile
   (c) 100,000,000 Angstroms = 1 centimeter
   (d) 1 foot = 1/6 fathom
   (e) 1 foot = 2/33 rod
   (f) 1 inch = 2.54 centimeters

6. Fill in the following blanks:

   (a) 12 feet = _____ inches
   (b) 1/88 mile = _____ yards

(c) 12,000 fathoms = _____ feet

(d) 1,000 Angstroms = _____ meter

(e) 1 meter = _____ feet

(f) 1 inch = _____ Angstroms

2.4 Successive Approximations to a Length-Measure

The two approaches to the measurement of length given in Chapter 1 and in Chapter 2 resulted in an inequality for the length measure of a given length. Neither of these gave a method for actually finding the exact value of the length measure.

The two procedures may be outlined as follows: Given a length of 2549 1/2 millimeters,

the method outlined in Chapter 1 would give an inequality which would use only counting number such as

$$2549 \leq d < 2550, \text{ in millimeter measure.}$$

The method developed thus far in Chapter 2 could give an inequality which would use numbers expressed in decimal notation, such as

$$2.549 \leq d < 2.550, \text{ in meter measure.}$$

An advantage of the second procedure is that it can be continued (at least theoretically) indefinitely. For example, if the length of 2549 1/2 millimeters is measured by using finer and finer divisions of the unit, the following inequalities might be obtained.

$$\left.\begin{array}{l} 2 \leq d < 3 \\ 2.5 \leq d < 2.6 \\ 2.54 \leq d < 2.55 \\ 2.549 \leq d < 2.550 \\ 2.5495 \leq d < 2.5496 \end{array}\right\} \text{ in meter measure} \qquad (2)$$

The number 2549 1/2 when written in decimal form would be 2549.5 and so the last inequality, $2.5495 \leq d < 2.5496$ does follow from what we know. These inequalities give estimates of d of steadily increasing accuracy. We also say that these inequalities give successive approximations to the meter measure d.

In practice, one would have to stop giving successive approximations when we reach the smallest division of length that can be counted. For example, in the last inequality the final 5 in the fourth decimal place

refers to 5 tenths of a millimeter. To obtain this, we need a very fine scale divided into tenths of a millimeter and probably a magnifying lens to count them. To continue that series of inequalities to the next decimal place would require an even more delicate method of measurement or some indirect method.

In practice, therefore, one is limited to a certain number of reliable decimal places in the inequalities.

Exercise 3

1. Which of the following is a more accurate estimate for the same length?

   (a) $3.25 \leq d < 3.26$ in meter measure or
   $3256 \leq d < 3257$ in millimeter measure

   (b) $561 \leq d < 562$ in centimeter measure or
   $56 \leq d < 57$ in millimeter measure

   (c) $4789 \leq d < 4790$ in millimeter measure or
   $4.789 \leq d < 4.790$ in meter measure

2. Which of the following gives the most accurate estimate of a certain length?

(a)	in meter measure	$5.81 \leq d < 5.82$
(b)	in kilometer measure	$0.0058 \leq d < 0.0059$
(c)	in millimeter measure	$5811 \leq d < 5812$
(d)	in centimeter measure	$581.14 \leq d < 581.15$

3. Suppose the dekameter measure d of a length satisfies the inequality $8.9674 \leq d < 8.9675$. Write inequalities for d in terms of the units used in Problem 2.

4. At what stage in the following series of approximations has the procedure of successive division been violated?

$$5 \leq d < 6$$
$$5.6 \leq d < 5.8$$
$$5.63 \leq d < 5.64$$
$$5.631 \leq d < 5.632$$

Does any one of these inequalities contradict another?

2.5 The Determination of the Length-Measure

Let us use Figure 6 to illustrate the inequalities of the preceding section. On each separate diagram the thick part of the line indicates the range of the inequality. To illustrate $2 \leq d < 3$, we make that part of the length which is between 2 and 3 thicker than the rest.

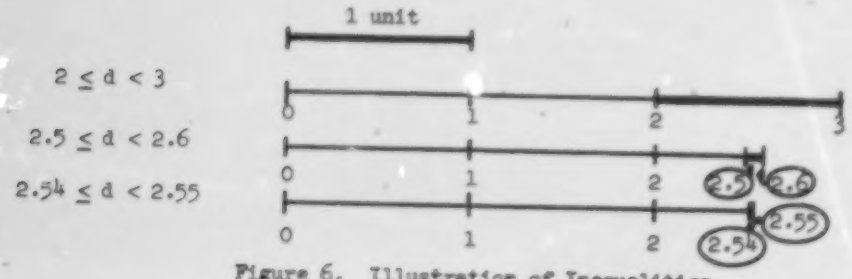


Figure 6. Illustration of Inequalities

It appears from this figure that the third successive approximation, with only three decimal digits, already gives a remarkably accurate representation of the length of the line. The next approximation could not be illustrated well because $2.5495 \leq m < 2.5496$ would just look like this:

The figure for the next approximation, say $2.54951 \leq d < 2.54952$, would look even more like a definite line with no "range of uncertainty" at the right-hand end.

From these diagrams, it appears that we could keep on getting closer to the exact measure of the length if we used smaller divisions of the unit. Of course, no matter how small the division, our measurement gives us only an inequality. For example, we might find that

$$2.549512 \leq d < 2.549513.$$

Consequently, with this division of the unit we have found a lower estimate 2.549512 and an upper estimate 2.549513 for the measure d.

However, if we keep on dividing the unit and measuring with this new division, we believe that this process will define a single number, a number which is between any lower estimate and any upper estimate. The single number that is defined by this process is the exact measure number of the length.

How is this measure number expressed? If we use a measuring process in which each division is one-tenth of the preceding division, the measure number will be obtained as a decimal in which the number of digits goes on and on. For example, suppose a length is exactly $\frac{10}{3}$ meters long, and we try to measure this length with a meter stick by continuelly dividing into tenths. Our successive estimates to this measure would read as follows:

$$3 \leq d < 4$$
$$3.3 \leq d < 3.4$$
$$3.33 \leq d < 3.34$$
$$3.333 \leq d < 3.334$$
$$3.3333 \leq d < 3.3334$$

and so on. If we did know that the sequence of estimates illustrated above would continue indefinitely, we would know that

$$d = 3.3333 \ldots$$

(Here the dots mean that the threes continue indefinitely.) Of course, this number d has the simpler name $3\frac{1}{3}$ or $\frac{10}{3}$ but this does not mean that every decimal has such a simple name.

To summarize: the measure of a length is assumed to be an exact number. This number may be a decimal which cannot be written exactly. However, we can always get estimates as close to this number as we please.

Exercise 4

1. Write down the first five successive estimate inequalities in decimal form, to the following length measures:

   (a) 49.3747921 (b) 8.999999
   (c) 3.22 (d) 4
   (d) $4\frac{1}{2}$ (f) $4\frac{1}{7}$

2. Guess the value of d in each of the following cases:

$$4 \leq d < 5$$
$$4.1 \leq d < 4.2$$
$$4.16 \leq d < 4.17$$
$$4.166 \leq d < 4.167$$

and so on,

$$2 \leq d < 3$$
$$2.6 \leq d < 2.7$$
$$2.66 \leq d < 2.67$$
$$2.666 \leq d < 2.667$$

and so on,

$$3 \leq d < 4$$
$$3.1 \leq d < 3.2$$
$$3.14 \leq d < 3.15$$
$$3.141 \leq d < 3.142$$
$$3.1415 \leq d < 3.1416$$
$$3.14159 \leq d < 3.14160$$

and so on.

Of which of your answers are you certain?

2.6 How Lengths Are Quoted in Practice

To simplify our discussions, we have sometimes considered lengths which can be given exactly by a measure number. An example of this is: 3.67, which contains only a few decimal digits. Actually, nearly every length is such that its measure is bound to be given by a nonterminating decimal. Since it is impossible to quote such a length exactly, we recognize that all statements about measurements imply an inequality and thus, two numbers. However, for convenience, we quote one number and make an agreement about what inequality should be implied by this number.

Before discussing this agreement, we must consider the topic of significant figures. We will define significant figures as figures which are the result of a measurement and not a result of conversion of units. Consider the examples of conversion of units in Section 2.3. We saw there that the length of 23.7 meters can be expressed as 2,370 centimeters or 23,700 millimeters. In these two cases, the new measures were not found by the act of measuring. They were merely arrived at by converting from meters to centimeters or from meters to millimeters. Then the only digits in any of these measures which were the result of measuring were the 2, the 3 and the 7. The zeros in the second and third numbers were introduced as the result of converting units and not as a result of measuring. We say that only three figures are significant in any of these numbers. Of course, if you see a length such as 23,700 mm, with no additional information, you cannot tell

how many digits are significant. When we discuss scientific notation, we shall learn a form for writing measure numbers which will clearly show how many figures are significant.

Let us return to the question of what inequality is implied by giving a single number for a measure. Suppose, as the result of a series of measurements, we have obtained the following successive approximations:

$$2 \leq d < 3$$
$$2.5 \leq d < 2.6$$
$$2.54 \leq d < 2.55$$
$$2.549 \leq d < 2.550$$
$$2.5494 \leq d < 2.5495$$

Looking at the second inequality, we can say for certain that this inequality shows that d is closer to 3 than to 2. Thus if, for the purposes of the measurement, it is enough to give d to one significant figure, then the value to give is 3 rather than 2.

Continuing, we can see that the third inequality shows that d is closer to 2.5 than to 2.6; thus, if we require two significant figures, 2.5 would be the best value to take.

To three significant figures, the best value for d is 2.55, and to four significant figures, the best value for d is 2.549.

You will have noticed by now that to know the best value for d at any stage of the approximation, it is necessary to know the next approximation. We shall make the following agreement or convention about stating measure numbers:

> At any stage of approximation, we use that estimate which is closer to the measure.

Now, if this convention is adopted as standard, we can make a definite statement about the possible error in any given measurement. Suppose a measure d is quoted as 2.55 to three significant figures; then it implies

$$2.545 \leq d < 2.555. \qquad (9)$$

If it were slightly less than 2.545, it would be closer to 2.54 than to 2.55, and so would be given as 2.54; similarly, if it were slightly greater than 2.555 it would be closer to 2.56 than to 2.55, and so would be quoted as 2.56. Thus, the possible error in the quoted value for the measure is no greater than .005. That is to say, the measure is given to the nearest hundredth (.01) and we are certain that this value is closer

than half of this unit to the accurate value. Therefore, we are certain that this number, 2.55, which is correct to three significant figures described any measure which could be reported by the inequality $2.545 \leq d < 2.555$.

Exercise 2

1. How many significant figures are there in each of the following numbers?

(a) 573.02
(b) 2.91
(c) 3.14159
(d) .005706
(e) 5,296,000
(f) 3.760

2. Quote the following measures to two significant figures:

(a) $34.06 \leq d < 34.07$
(b) $.0765 \leq d < .0766$
(c) $1374 \leq d < 1375$
(d) $.000567 \leq d < .000568$
(e) $29,783 \leq d < 29,784$
(f) $125 \leq d < 135$
(g) $.0195 \leq d < .0205$

3. Write the inequalities implied by the following measure statements:

m is approximately

(a) 2.6
(b) .075
(c) 2604
(d) 1.059
(e) .003
(f) 276.53

2.7 Exponents

In the discussion of the metric system for measuring lengths, we saw that the fundamental unit of length was the meter (m) but that for some purposes it may be convenient to use a smaller unit such as decimeter (dm), centimeter (cm), millimeter (mm), or a larger unit such as dekameter (dkm), hectometer (hm), kilometer (km). These units are related as follows:

$$1 \text{ m} = 10 \text{ dm} = 100 \text{ cm} = 1000 \text{ mm} = \frac{1}{100} \text{ hm} = \frac{1}{1000} \text{ km}.$$

We see that these new units are obtained from the meter either by successively dividing by 10 or successively multiplying by 10.

If we want to measure very small distances, such as the size of an atomic nucleus, we might have to use the unit that we would get by dividing the meter into 10 equal parts, and then dividing this part into 10 equal parts, and so on for 15 times. On the other hand, if we want to measure

very large distances such as the diameter of the galaxy, we would have to use a unit that we would get by taking a unit 10 times as large as the meter, and so on for 22 times. The ratio of the length of this unit to the length 1 meter would be the number written one followed by twenty-two zeros.

$$1 \times 10 = 10$$
$$10 \times 10 = 100$$
$$100 \times 10 = 1000$$
$$\underbrace{10 \times 10 \times \ldots \times 10}_{22 \text{ numbers } 10} = 10{,}000{,}000{,}000{,}000{,}000{,}000{,}000$$

Similarly, the ratio of the length of the small unit to the length 1 meter would be the number written as a decimal point followed by fourteen zeros and then by one.

$$\frac{1}{10} = 0.1$$
$$\frac{1}{10 \times 10} = \frac{1}{100} = 0.01$$
$$\frac{1}{10 \times 10 \times 10} = \frac{1}{1000} = 0.001$$
$$\frac{1}{\underbrace{10 \times 10 \times \ldots \times 10}_{15 \text{ numbers}}} = 0.000000000000001$$

It is clear that for problems involving such large or small numbers a better notation is needed.

Let us begin by considering the units of length larger than the meter. We may write:

$$1 \text{ m} = 1 \text{ m}$$
$$1 \text{ dkm} = 10 \text{ m}$$
$$1 \text{ hm} = 100 \text{ m} = 10 \text{ dkm}$$
$$1 \text{ km} = 1000 \text{ m} = 100 \text{ dkm} = 10 \text{ hm}$$

It is clear that the different numerical factors are obtained by successive multiplications with 10, that is

$$10 = 10 \times 1$$
$$100 = 10 \times 10 \times 1$$
$$1000 = 10 \times 10 \times 10 \times 1$$

Another way to write the first line of this would be

$$10^1 = 10 \times 1$$

where the first one tells how many factors of ten there are in the number. We read this: "Ten to the first power equals ten". The second line could then be written:

$$100 = 10^2 = 10 \times 10 \times 1$$

where the number 2 tells how many factors of ten there are in the number one hundred. We read this statement one hundred equals ten to the second power equals ten times ten times one. Using the same method, we note then 1000 is the product of three tens and could therefore be written 10^3 (ten to the third power).

A number such as 1, 2 or 3 which is written small and to the upper right side of the number 10 is called an exponent of ten. We shall use any counting number 1, 2, 3, 4, 5, ... as an exponent. The exponent of ten will count the number of times ten multiplies one. For example

$$10^4 = 10 \times 10 \times 10 \times 10 \times 1$$
$$10^5 = 10 \times 10 \times 10 \times 10 \times 10 \times 1$$

and so on. Thus, 10^{15} will equal one multiplied by the product of fifteen tens.

Sometimes we shall use zero as an exponent. Because of our definition of exponent, 10^0 equals one multiplied by ten zero times, that is, one multiplied by no tens, or one not multiplied by ten; consequently,

$$10^0 = 1.$$

Let us see if we can find another meaning for the exponent of ten. We have

$$10^0 = 1$$
$$10^1 = 10$$
$$10^2 = 100$$
$$10^3 = 1{,}000$$
$$10^4 = 10{,}000$$
$$10^5 = 100{,}000$$

and so on. Count the number of zeros on the right-hand side of these equations and compare this number with the exponent. Can you now explain why 10^{15} is the number one followed by fifteen zeros?

We can find still another interpretation of the exponent of ten. We can now say

$$200 = 2 \times 100 = 2 \times 10^2$$

or

$$3.57 \times 10^3 = 3.57 \times 1000 = 3,570.$$

This indicates that we can rewrite numbers in forms where the decimal point follows the first significant figure. Look at these expressions.

$$24.768 = 2.4768 \times 10^1$$

$$247.68 = 2.4768 \times 10^2$$

$$2476.8 = 2.4768 \times 10^3$$

$$24768. = 2.4768 \times 10^4$$

The decimal point in the numbers of the right hand column follows the first significant figure. However, in the left hand column we note that the decimal occupies a different position in each number. In the first number the decimal point is one place to the right, in the second number it is two places to the right and so on. Looking at the left hand column, do you see that the exponent of ten counts the number of places the decimal point is to the right of the first significant figure?

For example, in the third equation the factor 10^3 means the decimal point placement is three places to the right, that is the decimal point has been displaced from after the digit "2" to after the digit "6". In the same way, a factor 10^0 should mean that the decimal point is not displaced; therefore, we should have

$$2.4768 \times 10^0 = 2.4768.$$

Does this agree with our previous statement that $10^0 = 1$?

We can use this interpretation of the exponent of ten for finding the position of the decimal point in decimal form when the number is expressed in terms of a factor times some power of ten. We know that a number such as 247 can be written 247., with a decimal point following the seven. In fact, a decimal point may be considered as following every counting number. Thus, the rule for the placement of the decimal point does apply to the multiplications

$$247 \times 10^1 = 2470$$

$$247 \times 10^2 = 24700.$$

Using this notation, we may write

$1\ \text{dkm} = 10^1\ \text{m}$, $1\ \text{hm} = 10^2\ \text{m}$, $1\ \text{km} = 10^3\ \text{m}$,

to change length measures from one unit to another. For example, a length

$$2.4768\ \text{km} = 2.4768(10^3\ \text{m}) = 2.4768 \times 10^3\ \text{m}$$

$$= 2476.8\ \text{m}.$$

To summarise: The number ten with the exponent n, where n = 0, 1, 2, 3 ... , is written 10^n. It equals the number one followed by n zeros or, what is the same, the number one multiplied by the product of n tens. If a number is multiplied by 10^n, the effect is the same as moving the decimal point n places to the right, all empty places being filled by zeros.

Exercise 6

1. Write these numbers in a form that does not contain exponents:

(a) 63.475×10^2

(b) 2.3×10^4

(c) 0.0004×10^3

(d) 10.562×10^0

(e) 10^6

(f) 3.008×10^1

(g) 16.0×10^3

(h) 5.280×10^4

(i) 10×10^2

(j) $1 \times 10^2 \times 10^3$

2. Write these numbers in a form using exponents in such a way that the decimal point follows the first significant figure:

(a) 6,400,000

(b) 6,475

(c) 4.56

(d) 314,159

(e) 3,000,000,000

(f) 256

(g) 9,327.560

(h) 98.763

3. Which of the following are correct?

(a) $4 \times 10^3 \times 5 \times 10^2 = 20 \times 10^5$

(b) $7 \times 10^1 \times 4 \times 10^5 = 11 \times 10^6$

(c) $2 \times 10^3 \times 3 \times 10^2 = 6 \times 10^5$

(d) $19 \times 10^0 \times 7 \times 10^3 = 133 \times 10^3$

(e) $10^2 \times 10^3 \times 10^0 = 10^5$

(f) $10^2 \times 10^3 \times 10^1 = 10^5$

(g) $3 \times 10^2 \times 5 \times 10^3 = 1.5 \times 10^5$

(h) $1.2 \times 10 \times 1.2 \times 10^3 = 1.44 \times 10^5$

(i) $10^0 \times 2 \times 10^0 = 2 \times 1$

(j) $11 \times 10^2 \times 2.56 = 2.816 \times 10^4$

(k) $160 \times 2.5 \times 10^0 = 4 \times 10^2$

(l) $5 \times 10 \times 2 \times 10 = 10^3$

4. Fill in the blanks:

(a) 2.3456 km = _____ cm

(b) 2.4 km = _____ m

(c) 0.000064 m = _____ cm

(d) 5.62 cm = _____ m

(e) 37.6 mm = _____ km

(f) 37.76×10^2 cm = _____ mm

(g) $.057 \times 10^2$ m = _____ mm

(h) 3,762,598 mm = _____ km

(i) 4.578×10^5 cm = _____ km

(j) 4.67×10^6 m = _____ cm

(k) 2.58×10^0 cm = _____ mm

(l) $.0057 \times 10^3$ m = _____ mm

2.8 Negative Exponents. Scientific Notation

What can we do with the units smaller than a meter? We have

$$1 \text{ m} = 1 \text{ m}$$

$$1 \text{ dm} = \frac{1}{10} \text{ m}$$

$$1 \text{ cm} = \frac{1}{100} \text{ m} = \frac{1}{10} \text{ dm}$$

$$1 \text{ mm} = \frac{1}{1000} \text{ m} = \frac{1}{100} \text{ dm} = \frac{1}{10} \text{ cm} .$$

Here the different numerical factors are obtained by successive divisions by 10, or, what is the same, by successive multiplications by $\frac{1}{10}$. We may therefore write

$$1 \text{ dm} = \frac{1}{10^1} \text{ m}, \; 1 \text{ cm} = \frac{1}{10^2} \text{ m}, \; 1 \text{ mm} = \frac{1}{10^3} \text{ m}$$

because $10^1 = 10$, $10^2 = 100$ and $10^3 = 1000$.

Again, these relationships may be used to convert units into one another. For example,

$$37.8 \text{ mm} = 37.8\left(\frac{1}{10^1} \text{ cm}\right) = 3.78 \text{ cm}$$

$$37.8 \text{ mm} = 37.8\left(\frac{1}{10^2} \text{ dm}\right) = .378 \text{ dm}$$

$$37.8 \text{ mm} = 37.8\left(\frac{1}{10^3} \text{ m}\right) = .0378 \text{ m} .$$

To obtain these results, we used the following:

$$37.8 \times \frac{1}{10^1} = 3.78$$

$$37.8 \times \frac{1}{10^2} = .378$$

$$37.8 \times \frac{1}{10^3} = .0378 .$$

These examples suggest that multiplying by $\frac{1}{10^1}$ or, what is the same, dividing by 10^1, is equivalent to a displacement of the decimal point one place to the left. Similarly, multiplying by $\frac{1}{10^2}$ displaces the decimal point two places to the left and multiplying by $\frac{1}{10^3}$ or dividing by 10^3 displaces the decimal point three places to the left.

We have seen that the decimal point is shifted either three places to the right or three places to the left accordingly as we multiply or divide by 10^3. For example,

$$3{,}287.6945 \times 10^3 = 3{,}287{,}694.5$$

$$3{,}287.6945 \div 10^3 = 3.2876945 .$$

It is useful to be able to write both of these equalities as multiplications. To do this, we agree that the exponent 3 means displacement of the decimal point to the right when multiplication is indicated. Let us also agree, that if the displacement of the decimal point is to be to the left, we will express the exponent as -3 which is read "negative three". Thus,

$$3{,}287.6945 \times 10^{-3} = 3.2876945$$

and

$$123.6 \times 10^{-3} = .1236$$

and

$$42 \times 10^{-3} = .042$$

because the displacement of the decimal point to the left leaves an empty place which must be filled by a zero.

Why can we make such an agreement? We can do it because the symbol 10^{-3} did not have a meaning before and therefore we can make it mean anything we please. Of course, once we make such an agreement, we must accept all its consequences. For example, we know that multiplying by $\frac{1}{10^3}$ will also shift the decimal point three places to the left; consequently, we must agree that

$$10^{-3} = \frac{1}{10^3} = \frac{1}{1,000} .$$

Similarly,

$$10^{-1} = \frac{1}{10}, \quad 10^{-2} = \frac{1}{100} ,$$

and so on.

Let us summarize our agreement about the meaning of a negative exponent: A number such as 10^{-n}, where n = 1, 2, 3, ... , equals one divided by 10^n or, one divided by the number which is one followed by n zeros. The result of multiplying a number by 10^{-n} is to move the decimal point n places to the left.

For example:

$$10^{-5} = \frac{1}{10^5} = \frac{1}{100,000}$$

$$237.4 \times 10^{-5} = .002374.$$

We may use the notation of negative exponents to express divisions of a meter in terms of a meter. We see that

$$1 \text{ dm} = 10^{-1} \text{ m}, \quad 1 \text{ cm} = 10^{-2} \text{ m}$$
$$1 \text{ mm} = 10^{-3} \text{ m} = 10^{-2} \text{ dm} = 10^{-1} \text{ cm}$$

and so on. The previous example, in which we changed 37.8 mm into other units, now can be written as follows:

$$\begin{aligned} 37.8 \text{ mm} &= 37.8 \times 10^{-1} \text{ cm} = 3.78 \text{ cm} \\ &= 37.8 \times 10^{-2} \text{ dm} = .378 \text{ dm} \\ &= 37.8 \times 10^{-3} \text{ m} = .0378 \text{ m} . \end{aligned}$$

Powers of ten may be used to write large or small numbers in a simple fashion. For example, the speed of light, which is 186,000 miles per second, can be written as 1.86×10^5 miles per second because

$$1.86 \times 10^5 = 186,000.$$

This new notation also emphasizes that only the three digits 1.86 are significant, that is, only these digits are the result of measurement. In

fact, more accurate measurements of the speed of light show that the speed is approximately 186,324 miles per second.

A numeral such as 1.86×10^5 in which the decimal point appears after the first non-zero digit is said to be written in scientific notation. The following are examples of numbers in scientific notation:

$$3.65 \times 10^2$$
$$5.28 \times 10^3$$
$$1.6 \times 10^{-3} \ .$$

The following are not in scientific notation: 36.5×10^1, 0.528×10^4, 16×10^{-4}.

Exercise 7

Express in scientific notation. Use a World Almanac or similar reference book to get the appropriate numbers where they are needed.

1. The speed of light correct to two significant digits.
2. The speed of light correct to five significant figures.
3. The number of Angstrom units in one centimeter.
4. The number of meters in one millimeter.
5. The number of kilometers in one millimeter.
6. The annual budget of the United States.
7. The measure of the area of the United States in square miles.
8. The measure of the area of your state in square miles.
9. The ratio of the number in Problem 7 to the number in Problem 8.

Carry out the indicated calculations and write the answer in scientific notation.

10. $(1.6 \times 10^3) \times (1.1 \times 10^2)$
11. $(1.6 \times 10^3) \times (1.1 \times 10^{-2})$
12. $(1.6 \times 10^{-3}) \times (1.1 \times 10^2)$
13. $(1.6 \times 10^{-3}) \times (1.1 \times 10^{-2})$
14. Use the facts that 36 in = 1 yd and 2.54 cm = 1 in to express the measurement 1 cm in yards.
15. If the speed of light is 1.86×10^5 miles per second, change it to meters per second. Use 1 mile = 5280×12 in and 1 in = 2.54 cm.

2.9 The Number Line

Figure 6 illustrated graphically some inequalities, and in Section 2.5 we considered what happens as the successive inequalities narrow down the range within which a number d lies. We said that a definite number is defined by such a process.

This definite number can be represented by a point on the lines in Figure 6, not by a shaded area which has a length to it, but by a single point. In this way we build up the idea of a line of numbers, or a number line, on which any number can be represented by a point.

This idea of a number line is closely related to the idea of length and to length measure, since all the operations which we have already met in connection with length, such as addition and comparison, are matched by similar operations with numbers on the number line.

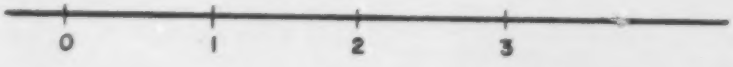


Figure 7. Number Line

To construct a number line, start with a line and mark one point on it with a zero. Lay off the counting numbers at equal spacings as illustrated in Figure 7. The spacing between zero and one is, of course, arbitrary, in other words, it can be chosen as any convenient length.

The point on the number line which was marked with a zero is called the origin and the point which was marked with one is called the unit point. The arbitrary spacing between the origin and the unit point is the unit distance. All of the numbers which are not counting numbers also have points assigned to them. For example, the point half way between the unit point and the point assigned to the number two is assigned the number $1\frac{1}{2}$.

As we have seen in Section 2.5, the number line also contains points such that the measure of their distance from the origin cannot be written as an exact decimal. Thus, the number $\frac{1}{3} = 0.3333 \ldots$ cannot be written as an exact decimal. It also happens that there are numbers which can neither be written as an exact decimal or as a simple fraction whose numerator and denominator are both integers. For example, π (the ratio of the circumference of a circle to its diameter) and $\sqrt{2}$ (the number which multiplied by itself produces 2) are numbers of this type. These numbers are marked approximately in Figure 9.

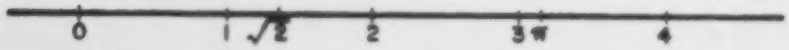


Figure 9. Number Line

The number line has the following important properties:

1. The number line orders numbers: if $a < b$, then the point representing a will be to the right of b.
2. Addition of numbers can be represented on the number line as an addition of lengths. It is left to you to see how this can be done.

Summing this up, we can say that every positive number can be represented by a point on the number line. Furthermore, the properties of the number line are closely related to the properties of length-measure.

Exercise 8

1. Draw your own number line to cover the first ten counting numbers. Mark the following numbers on it as accurately as you can:

$$2\tfrac{1}{2},\ 9.5,\ 8.6,\ 1\tfrac{1}{3},\ 5.25,\ 6.375,\ 4\tfrac{1}{14}.$$

2. On another number line, making each unit 10 cm in length, mark the following numbers:

$$1\tfrac{1}{3},\ 1.26,\ 1\tfrac{2}{7},\ 1.3,\ 1\tfrac{3}{11},\ 1.2.$$

Arrange these numbers in ascending order of magnitude.

3. Illustrate $2 + 3 = 5$ on a number line.

4. On the number line mark the points representing the following numbers:

$$\tfrac{1}{2},\ \tfrac{3}{2},\ \tfrac{5}{2},\ \tfrac{7}{2},\ \tfrac{9}{2},\ \tfrac{11}{2},\ \tfrac{13}{2}.$$

5. (a) On the number line mark the points representing the following numbers:

$$\tfrac{1}{3},\ \tfrac{2}{4},\ \tfrac{3}{5},\ \tfrac{4}{6},\ \tfrac{5}{7},\ \tfrac{6}{8},\ \tfrac{7}{9},\ \tfrac{8}{10}.$$

(b) Which of these rational numbers in part (a) can be written as exact decimals?

6. If it is possible, express each of the following numbers as a simple fraction whose numerator and denominator are both integers.

(a) .25
(b) .333 ...
(c) π
(d) .125
(e) $\sqrt{2}$
(f) .8333 ...
(g) .777 ...
(h) $\sqrt{4}$
(i) .363636 ...
(j) .454545 ...

(Note: In parts (i) and (j) the 3 dots mean that the established pattern of digits is repeated indefinitely.)

7. Express each of the following as an exact decimal wherever possible.

(a) $\frac{1}{8}$
(b) $\frac{3}{7}$
(c) $\frac{9}{20}$
(d) $\frac{7}{12}$
(e) $\frac{5}{11}$
(f) $\frac{5}{6}$
(g) $\frac{23}{16}$
(h) $\frac{2}{13}$

2.10 Summary

An opportunity to gain a clearer understanding of the nature of measurement and the role of units was given. The experiment on the unmarked stick gave a chance to take an arbitrary unit and divide it into equal smaller units and perform measurements with it. In addition, the need for conversion of units could be recognized since neighbors performing the same experiment were using sticks of different lengths. This parallels the situation in the world where different nations use different standard units of measurement. The length measure of the metric system was developed and the conversion of standard units was examined.

The problems of precision and error in measurement were made apparent. Scientific notation was introduced and its advantages in dealing with approximations and significant figures were explored.

The number line was introduced with the arbitrary choice of unit length. Then the correspondence of the points on the line and numbers was made. This provides us with a pictorial model of our number system.

Chapter 3
RELATIONS, FUNCTIONS AND GRAPHING

3.1 Introduction

For several years now you have been learning more and more in school about the idea of number. In the work that follows we shall have much to do with things -- objects that we can see, feel, hear, or, in general, know about with our senses. People who spend their lives studying numbers and number-like concepts are called mathematicians. Mathematicians are curious mostly about inventions of the mind. People whose life work deals with explaining things and events in nature are called scientists. Scientists are curious about the world of the senses and how to make it understandable. You might think that the mathematician and the scientist are doing completely different things and that they would have nothing useful to say to each other, but that is not true. There is a wonderful and mysterious connection between the thing world and the number world. We wish to explore this connection.

Because we must make a connection in our thoughts between the thing world and the number world, it is helpful to have an orderly way of describing how we can assign numbers to things. Also, we need to keep track of what we have done.

A little experiment will show the need for being careful in our record-keeping and will lead to a new idea that you will use again and again as long as you study mathematics. You probably have several books in your desk. Take some time out from your reading and get them out on top. Now, imagine that you are going to write a letter to a cousin telling how many pages each book has. Get a piece of paper and put the information down as briefly as you can.

Now let's examine the results. Did you really get the message across? Maybe you put down something like the following example:

Example A: 215 pages, 166 pages, 301 pages, and 277 pages.

If so, how is your cousin to know what books you have, or which book has how many pages?

Is the following example better?

Example B: English, Spanish, mathematics, social studies, 215, 166, 301, 277.

We agree that there is a bit more information here, but what book gets paired with what number of pages? Does the English book have 215 pages, the Spanish book, 166 pages, and so on? If that's what we mean to say, what would be a safe way of saying it? Perhaps you have foreseen these dangers and have written carefully

Example C:

"The English book has 215 pages,
the Spanish book has 166 pages,
the mathematics book has 301 pages, and
the social studies book has 277 pages."

In this case you have done a fine job and we could only suggest that to save pencil lead you prune it down a bit without losing any important information. Something like this:

Example D:

English	215
Spanish	166
mathematics	301
social studies	277

Do you agree that this little table contains everything we want to say? At first you may be tempted to say it does, but think for just a minute. Your cousin may have been working in the library counting books. In this case he might think the table says that there were 215 English books in the library. On the other hand he may have been helping the principal assign students to classes. Now he might understand the table to say that 277 students are enrolled in social studies.

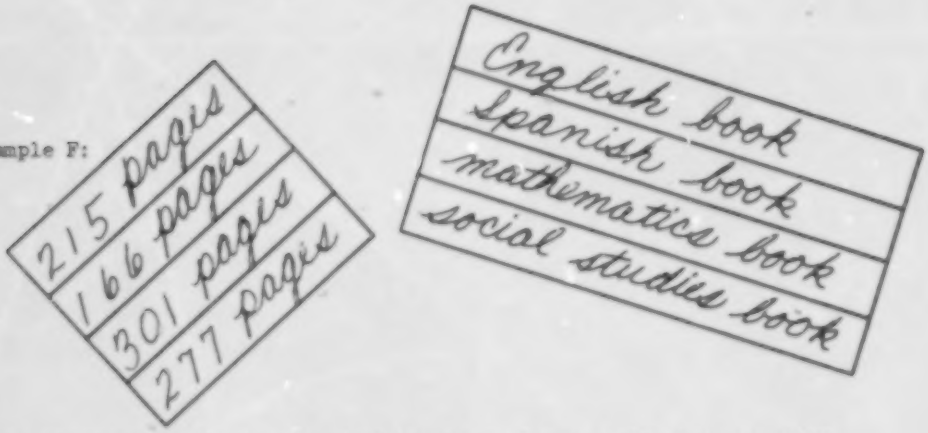
A more careful way of displaying the table might be like this:

Example E:

English book	215 pages
Spanish book	166 pages
mathematics book	301 pages
social studies book	277 pages

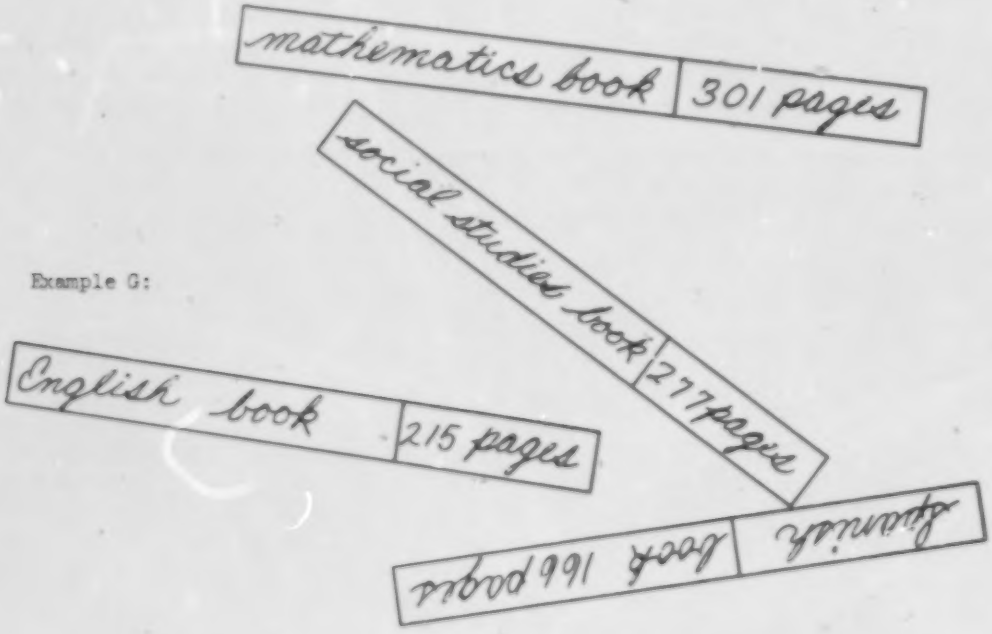
Could we take a pair of scissors and cut our table down the middle into two pieces without at the same time destroying some information?

Example F:



What has gone wrong? Do you see that we can no longer tell <u>for sure</u> which number of pages is attached to which book? This kind of cutting apparently does serious damage to the information in the table.

Try slicing the table another way, this time along each horizontal line instead of along the vertical line. We then have quite a clutter.



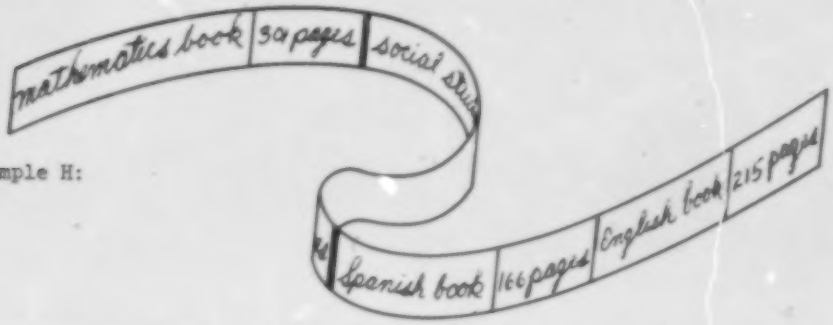
Example G:

Your cousin would complain justly that you had become an even messier letter writer than usual if you mailed these slips of paper to him--but he could

have no serious doubt about which book had how many pages. In other words, the information in the table has not been destroyed even though the paper on which it has been written has been badly slashed.

Gluing the strips together to form a long ribbon does not change the information at all, just the physical appearance. Notice that the order in which we glue the strips does not change the information in any way either. We can still learn from it,

Example H:



for example, that the Spanish book has 166 pages.

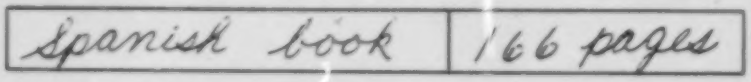
But if we should really go wild with the scissors again and chop up the tape into bits along each vertical division,

Example I:

all we get is confetti. The message about matching has again been destroyed.

3.2 Ordered Pairs

Looking back at our various attempts to convey our message, we see that examples C, E, G and H have been successful, whereas examples A, B, D, F and I have been failures. What brings success, and what leads to failure? The examples C, E, G and H are similar because the idea of pairing is strong in each. The symbol



or something like it, is the common theme. We will deal with such symbols so often that a simpler way of writing them will be useful. We choose the following way that is easy to typewrite and print:

(Spanish book, 166 pages)

We refer to this as an ordered pair and say the "Spanish book comma 166 pages" or "Spanish book, 166 pages" when reading aloud to someone.

By an ordered pair we mean a set containing exactly two elements in which one element is recognized as the first. For example, in the ordered pair, (Spanish book, 166 pages), Spanish book is the first element, and 166 pages is the second element. Since we read from left to right, picking out the first element is easy.

It is easy to see that example C is nothing more than a set of ordered pairs (or class of ordered pairs, or collection of ordered pairs, if you prefer). The same is true for examples C, D, E and H, but each example has its own characteristics. Example E is neat but C is wordy, D is incomplete, G is messy and H is a bit peculiar. However, they are all ways of writing down the same set of ordered pairs.

The set of ordered pairs which we have been discussing here is merely an example. There are many other examples of sets of ordered pairs which could be given. In fact, whenever there is a natural relation between two sets of things (or numbers) there is a corresponding set of ordered pairs. For example, here is a set of ordered pairs:

{ (Austin, Texas), (Annapolis, Maryland)
(Sacramento, California), (Albany, New York),
(Denver, Colorado), ... }

Can you see what the relation between the first and second elements of these pairs is? Can you add more pairs to this set? What first element belongs

with

(..., Illinois)

in this set? What second element belongs with

(Columbus, ...)?

Here is another set of ordered pairs

{(Phoebus, Mars), (Titan, Saturn), ...}

Can you add more pairs to this set? What pair in this set would have "Earth" as its second element?

3.3 Relations

Why is a set of ordered pairs convenient? It is convenient because it indicates an act of matching or relating. Our collection of ordered pairs relates 166 pages to Spanish book, 301 pages to mathematics book, 215 pages to English book, and 277 pages to social studies book. If we wished to, we could reverse the order of our ordered pairs. But if we agree to have the name of the book as the first element of each ordered pair and the number of pages as the second element, we avoid the confusion of having some of us saying it one way and others another way.

A set of ordered pairs will be called a relation. The relation that we have found between pages and books is displayed in the following set of ordered pairs.

(English book, 215 pages)
(Spanish book, 166 pages)
(mathematics book, 301 pages)
(social studies book, 277 pages)

Notice that this set is the one which appeared in Example E. When we manufactured F by chopping E down the middle, we destroyed a relation but were left with two sets which it is extremely useful to give special names. The set on the left, which we rewrite here horizontally, as follows:

{English book, Spanish book, mathematics book, social studies book}

is the set of first elements of the ordered pairs in the relation. It is called the domain of the relation. Note that mathematicians use braces, { }, to list the elements of a set.

The other set

{215 pages, 166 pages, 301 pages, 277 pages},

the set of second elements, is called the range of the relation. Notice that the order of listing the elements between the braces doesn't matter. We could just as well have said that the range was the set

{301 pages, 166 pages, 277 pages, 215 pages}.

We can now say that a relation matches each element of its domain to one or more elements of its range. Of course, we can't say precisely how this is done without looking at its ordered pairs. The relation we have been discussing has its domain filled with things and its range with numbers, but it is possible to have relations that relate things to numbers, or things to things, or numbers to numbers. As you might expect, mathematicians are especially interested in the last kind.

Exercise 1

1. Make a set of ordered pairs by assigning to each month of a leap year the number of days it contains.
   (a) How many ordered pairs are in the set?
   (b) How many elements are in the domain of the relation?
   (c) Write the set of elements in the range of the relation.
   (Hint: In a set, an element is listed only once.)
   (d) How many elements are in the range of the relation?
2. Suppose we had used the year following a leap year in Problem 1. Would we have answered Problems 1(a - d) any differently? Why?
3. The domain of a certain relation is the set {7, 10, 22, 1}; the range is the set {B, C, A, D}. From this information form one relation having five ordered pairs and one relation having seven ordered pairs.
4. Use the index in the back of one of your texts to obtain ordered pairs.
   (a) List four ordered pairs where different elements of the domain share the same element of the range.
   (b) List four ordered pairs where different elements of the range are assigned to the same element of the domain.
5. Given the following sets of ordered pairs:
   (a) {1, 2}, {4, 2}, {5, 2}, {6, 3}, {7, 3}
   (b) {0, 0}, {1, 0}, {2, 0}, {3, 0}
   Determine the domain and range.

3.4 An Experiment

You have probably heard it said that "a picture is worth a thousand words". We know that a relation is a set of ordered pairs. Some relations, for example, may contain such a large number of ordered pairs that it would be physically impossible to list every ordered pair in the set. It might, then, be a good idea to look around for other methods of displaying these ordered pairs.

Suppose we consider an interesting experiment. Our problem is this: "If a stack of identical books is placed at the edge of a table, what is the largest amount of overhang we can get from them before the books topple?"

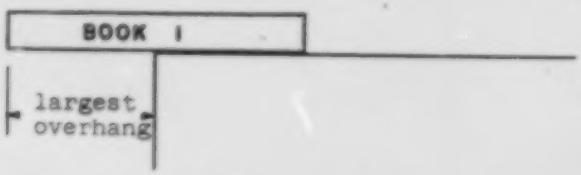


Figure 1

Let us suppose the length of each book is 24 cm . If we have only one book in our stack, it is easy to answer the question. A little experimentation will soon show us that we can move the book lengthwise beyond the table's edge until we have a largest overhang of 12 cm measured to the nearest centimeter.

This problem of balancing a stack of identical books and determining the largest amount of overhang suggests a question:

"What is the relation between the number of books in a stack and the largest possible overhang from this number of books?"

The first ordered pair in this particular relation then will be (1 book, 12 cm). This notation promises to be a little more bulky than we might like to have it. We have already seen that the elements of domain are books and the elements of the range are centimeters. We could strip these elements of their labels and write the first ordered pair as (1,12). Remember, however, if we do not label each individual element of the domain and range, we must state definitely from which set these elements are drawn.

How do we find the maximum overhang for the top book if there are two books in the stack? Remember that the books are the same size. If the second book is placed directly on top of the first book, the books are balanced. If we extend the top book out beyond the first book, we would find that the stack

of two books would topple off the table. From this it might appear that 12 cm is the maximum overhang that we can get, no matter how many books we have in the stack.

Let us pause a moment and see if we can't be a little more clever. Suppose we were to hold the first book firmly in the balanced position. Now balance the second book on the first book in the same way you balanced the first book on the table. But of course this isn't fair, since the stack would again topple if you were to remove your hand from the first book. But what would happen if we were now to slide the two books together back on the table until this stack of two books balanced? Would we have to push them so far that the bottom book would rest entirely on the table? If you have tried this, you know that the answer is no. You should find that a stack of two books will balance something like this.

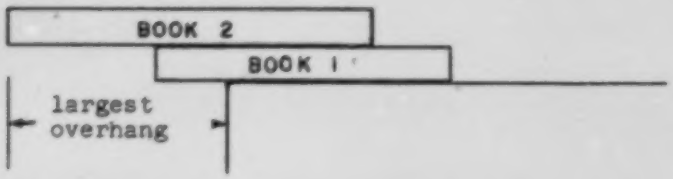


Figure 2

When we balance the books this way and measure the largest overhang, we might find that the distance is 18 cm to the nearest centimeter. We would then indicate this second act of balancing by the ordered pair, (2,18).

Now things are getting a bit more interesting. What happens when we add a third book to the stack and what must we do to ensure maximum overhang in this case? Let us use the same method that already worked so well for us. Hold these two books firmly in position and balance the third book on top of the second book in the same way you balanced the first book on the edge of the table. Does this give you a largest overhang for the third book? Does it give you a largest overhang for the two top books? Taking these two books as a unit, we note that the overhang is too great to give us a balance. Move this unit of two books back on the bottom book until they just balance on the bottom book. You should recognize that we have actually balanced these two books on the bottom book in the same way we balanced two books on the edge of the table.

All three books are now balanced as a unit. Our problem now is to balance this unit on the edge of the table in such a way that they will give us a greatest overhang. The same procedure we have already used should also

work in this case. Move this unit back onto the table until it just balances. Now we get a balanced stack of three books.

You might doubt that this method is the one which gives a greatest overhang. If you do doubt it, try other ways you might think of and see if you can get a greater overhang. We haven't tried to explain why this act of balancing works the way it does. This, of course, suggests that finding a way of solving a problem is not enough in science and mathematics. We must also be able to explain why this method works. At this point, however, we are not ready to explain why this particular approach works. Let us just say that it does work.

With the three books balanced, we might find that the measure (of the overhang) to the nearest cm is 22 cm. The result of this balancing could then be recorded as an ordered pair, (3,22).

We won't go into a detailed description of how to balance four books or five books in order to get a largest overhang (Figure 3). If you think carefully, you can probably find a way to do this. One thing which might make this an interesting experiment to perform is that very soon you can get a stack of books balanced in such a way that the top book is not over the table at all.

Figure 3

3.5 Graphing of Ordered Pairs

Suppose that the set of ordered pairs we collected in this experiment was

$$\{(1,12), (2,18), (3,22), (4,25), (5,27)\}.$$

The first element of each ordered pair tells us how many books there were in the stack, and the second element refers to the centimeter measure of the maximum overhang from the edge of the table.

The set notation, { }, has given us a handy method for listing the ordered pairs of a relation. However, sometimes a simple listing of these ordered pairs does not give us the best idea about all features of a relation.

Let us then consider other methods of showing the ordered pairs of a relation. We have already had experience with the number line where every point has a definite real number as its coordinate and each real number is a coordinate of a single point. We shall relate the coordinates of a number line to the magnitude of various physical measures. We have an example of this in Section 2.1 where we related the coordinates on a ruler to length. We could also have a number line for area, volume, mass, time or other physical quantities.

Relations in which the ordered pairs are pairs of numbers may be displayed on two number lines. For convenience, let us choose two lines which are parallel. These lines can be horizontal or vertical or even diagonal; it does not matter. It is important to know which line we let represent the domain and which line represents the range. We can take care of this easily by giving each line an appropriate label. We should also indicate the coordinates of some of the points on each of these lines. Our coordinate lines might look like this,

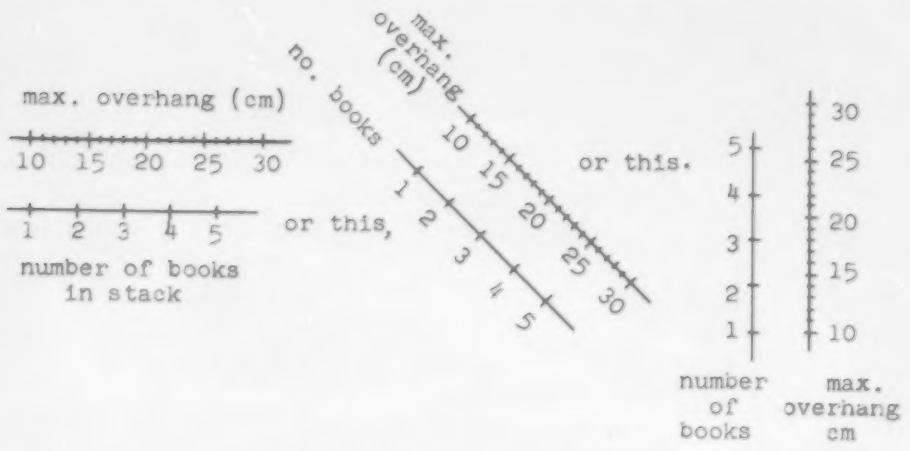


Figure 4

In this example, let us choose a pair of vertical number lines. For convenience, let us put the elements in the domain on the line to the left and

the elements in the range on the line to the right.

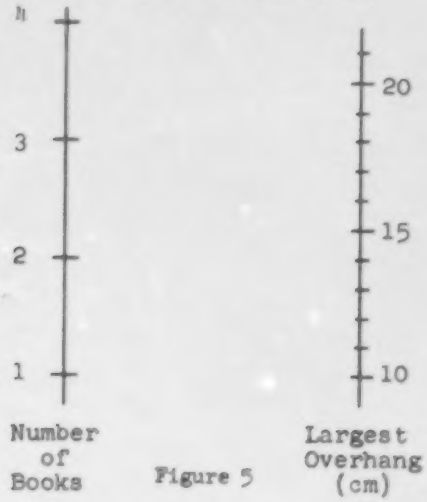


Figure 5

As we have already stated, we are interested in displaying the set of ordered pairs in the relation. We do this by drawing an arrow from each element in the domain to the corresponding element in the range. Remember, the labels on the number lines only give us an idea of the kind of things in the domain and the range. Now the arrows indicate which pairs are in the relation. The arrowheads indicate which set is the range, and the tails indicate which set is the domain.

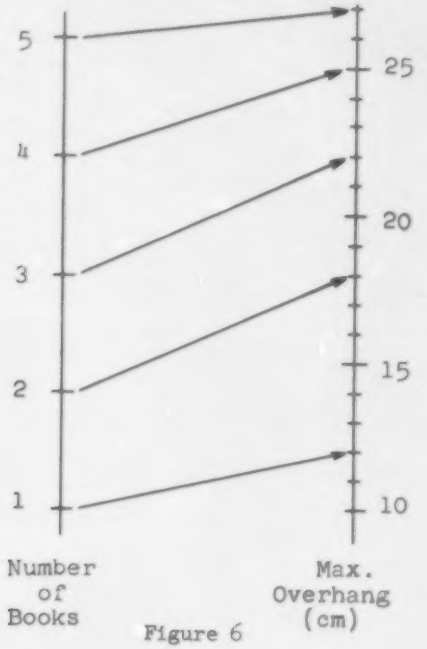


Figure 6

This diagram then is a graph of the ordered pairs in our relation.

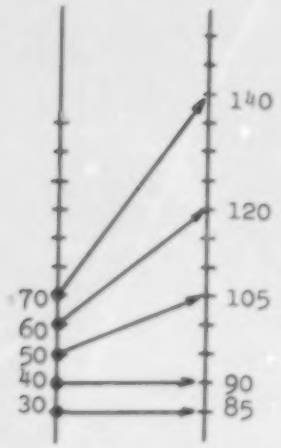
Exercise 2

Graph (by drawing arrows from one number line to another) each of the following sets of ordered pairs.

1. {(1,1), (2,3), (3,6), (4,10), (5,15)}
   (a) What are the elements of the domain?
   (b) What are the elements of the range?
   (c) If a relation contained the above ordered pairs and if the number 6 were also an element in the domain of this relation, what would you guess for the corresponding element in the range if the same pattern were repeated?
2. {(1,1), (2,2), (3,3), (4,4), (5,5)}
   (a) What are the elements of the domain?
   (b) What are the elements of the range?
   (c) Is the ordered pair (6,6) also an element in the set defined by this relation? Why?
3. {(1,3), (2,2), (3,3), (4,2), (5,3), (6,2)}
   (a) What are the elements of the domain?
   (b) What are the elements of the range?
   (c) If a relation contained the above ordered pairs and if the number 15 were also an element in the domain of this relation, what would you guess for the corresponding element in the range if the same pattern were continued?
   (d) If the number 8 were an element in the domain of a relation similar to that described in part (c), what would you guess for the corresponding element in the range if the same pattern were continued?
   (e) State a rule for writing ordered pairs belonging to a relation similar to parts (c) and (d) if the domain were the counting numbers 1 through 100.
4. {(3,1), (2,2), (1,3), (1,4), (2,5), (3,6)}
   (a) What are the elements of the domain?
   (b) What are the elements of the range?
   (c) If a relation contained the above ordered pairs and if the number 4 were also in the domain of this relation, what would you guess for the corresponding element in the range if the same pattern were continued?

Each of the following graphs, (5 - 10), defines a relation. List a set of ordered pairs which defines the same relation. List the domain and the range for each relation.

5.



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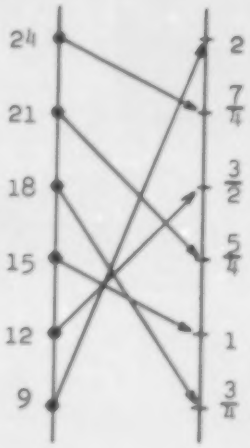

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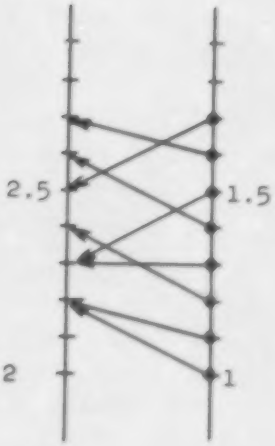
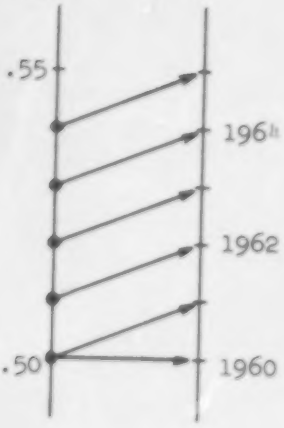
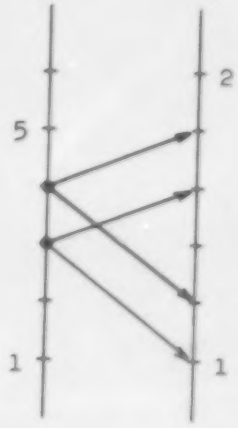


8.



9.



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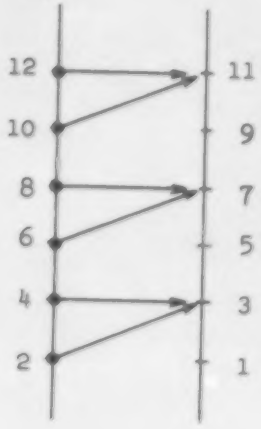


3.6 Functions

In the preceding section we had some experience with graphing relations on parallel number lines. We took careful note of which number line displayed the graph of the elements of the domain of the relation and which displayed the graph of the range of the relation. Let us look at these relations for a moment and see if there appears to be other properties that we can see in them. Consider for a moment the graphs in the preceding set of exercises.

How might we classify the graphs in Problems 5 - 10? One thing we could do would be to look for patterns established by the arrows, or we might compare the number of arrows displayed by any relation to the number of elements in the domain and range of the relation.

In Problem 5, we could note that there are 5 arrows. This says that the relation contains 5 ordered pairs. There are also 5 elements in both the domain and the range of this relation. Careful thought will tell us that this can work only if each element in the domain is used just once and each element in the range is used just once. Do any of the other relations have this same property of using each element in the domain just once and each element in the range just once in forming ordered pairs? Problems 7, 8, 9 and 10 certainly do not.

We can use this same approach now to see if there are differences in these last four graphs. Looking at Problem 7, we see that there are 8 elements in the domain, 6 elements in the range and 8 ordered pairs. The only way we could get 8 ordered pairs would be if each element in the domain were used just once, but some element (or elements) in the range were used more than once. In this case, 2 elements were used twice, namely, (1, 2.2), (1.1, 2.2) and (1.3, 2.3), (1.5, 2.3).

Would it be possible to have a relation with 8 elements in the domain, 8 elements in the range and 6 ordered pairs? Why not? If you recall our definition of a relation, you will remember that we said the domain is the set of first elements in the ordered pairs. This tells us that we must have at least as many ordered pairs in the relation as there are elements in the set we call the domain. The same line of reasoning would tell us that we must also have at least as many ordered pairs in the relation as there are elements in the set we call the range.

Now compare the graph in Problem 8 to that in Problem 7. Do they appear to have the property of using each element of the domain just once? We note that there are 5 elements in the domain, 6 elements in the range and 6 ordered pairs in the relation. This means that one element of the domain had to be used more than once in forming ordered pairs. Problem 9 follows this general pattern (2 elements in the domain, 4 elements in the range and 4 ordered pairs) with both elements in the domain being used twice. Problem 10 seems to follow a pattern similar to Problem 7 where each element in the domain is used just once in forming the ordered pairs of the relation while some elements of the range are used more than once. In a way,

this last distinction is similar to that made in Problems 5 and 6 because in these problems we also noticed that each element of the domain was used just once.

Let us now define a special kind of relation. We shall call this special type of relation a function. A function is a set of ordered pairs such that each element of the domain appears in one and only one ordered pair. This allows more than one element in the domain to correspond to a particular element in the range (see Problem 10). It is important to realize that this definition does not allow any single element in the domain to correspond to two or more elements in the range.

The set of all first elements of the ordered pairs is called the domain of the function, and the set of all second elements is called the range of the function.

According to our definition of a function, we note that the graphs in Problems 5, 6, 7 and 10 are the graphs of functions. Problem 8 is not a function because here the element .50 in the domain of the relation appears in two of the ordered pairs, (.50, 1960), (.50, 1961). Problem 9 presents us with a similar situation where both elements in the domain of the relation are each used in two ordered pairs and hence, this relation is not a function either.

Exercise 3

Define a function which is suggested by the phrases in Problems 1 - 7. Be sure to specify the domain and the range of each function.

1. {(1,2), (2,3), (3,4), (4,5), (5,6), ...}
   (Hint: The symbol, ..., used in this manner, means that the ordered pairs continue indefinitely according to the pattern suggested.)
2. {(1,1), (2,4), (3,9), (4,16), (5,25), ...}
3. Election returns
4. Area of triangles
5. People's first names
6. With each positive integer associate its remainder after division by 5.
   (Example: 1 + 5 = 0 + remainder 1, hence, (1,1))
7. Associate with each length of the diameter of a circle the length of the circumference of the corresponding circle.
8. The cost of mailing a letter is determined from the weight of the letter as follows: it is 5¢ per ounce plus 5¢ for any fraction of an ounce.

Does this describe a function? Why?

What is the domain?

(Note that the Post Office will not accept a first class mailing which weighs more than 20 pounds.)

What is the range?

Complete the following ordered pairs in this relation: (domain is given in ounces)

(3.7, _____), (5, _____), (19.2,_____)

3.7 More Graphing

Let us do another experiment which will give us an opportunity to develop graphing further.

In this experiment we will use an irregular, clear glass container, such as one which contained salad dressing, vegetable oil, or syrup. Place a strip of "magic mending" tape (this is the clear type on which you can write) along the length of the bottle from top to bottom and write "bottom" on the end of the tape at the bottom of the bottle. Place a small but visible mark along the edge of the tape at the point where the tape is lined up with the inside level of the bottom of the empty container. Later, when you remove the tape from the bottle, we will assign the number zero (0) to this point.

Two more pieces of equipment are necessary for this experiment. We need a unit of volume. A 22 cm^3 plastic pill bottle works very well for this purpose and these are easily available in most drug stores. For lack of a better name for the unit of volume let us call it the "glug". Secondly, we need a container of water whose volume is greater than that of the irregularly shaped container. It would probably also be a good idea to have some paper towels handy to lay on the table under the various containers being used in the experiment.

Submerge the pill bottle in the container of water so that when the pill bottle is removed from this container it is completely full of water. Carefully transfer this glug of water to the irregular bottle. Care should be taken to avoid spilling since we want to pour the same volume of water into the bottle each time. On the other hand, you should be cautioned to move right along. Have you ever observed how a waitress in a restaurant carries a cup of coffee without spilling it? She certainly doesn't waste time being too concerned about spilling, but at the same time she is reasonably careful not to spill.

As soon as you have poured the glug of water into the bottle, make another small mark on the tape at the level of the liquid in the bottle. When you sight along the level of the liquid, you may have some doubt about just where the liquid level is. You should be careful that your eye is sighting at the liquid level (Figure 7). If you sight down on the surface of the liquid or up from some point below this surface, you will notice that there is even more uncertainty as to where the mark on the tape should be made. When your line of sight is level with the surface of the liquid, you will probably notice three lines, any one of which could be identified as the surface level. The more distinct of these three lines will be the middle line, and let us agree to use this as the reference point for marking the tape.

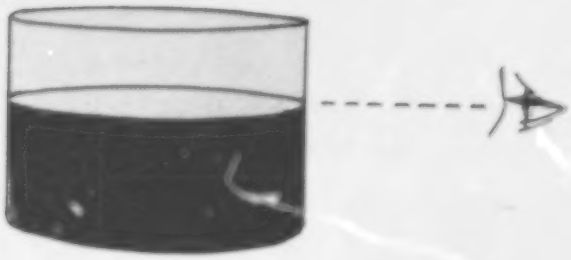


Figure 7

As soon as you have made this mark on the tape, make a mental note that the mark is related, in some way, to one glug of water.

Now repeat the process of filling the pill bottle with water and pouring this water into the bottle. Again make a small mark at the point on the tape which the surface of the liquid has now reached.

Carry on this procedure until you reach the point where it looks as though the addition of one more glug of water will result in overflow.

Now, take a clean sheet of notebook paper. On the paper draw a straight line with a ruler. Remove the tape from the bottle and place it on the paper in such a way that the edge of the tape on which the marks were made is lined up with the line which you drew on the paper (Figure 8).

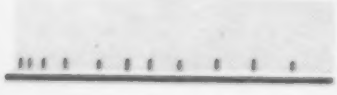


Figure 8

Write the number zero opposite the very first mark you made on the tape. Draw a second line on the paper to the left of the first line, parallel to the first line. Connect these two lines with a line segment which has the zero mark on the first line you drew as an end point and its other end point on the other line. Put an arrowhead on this line segment at the end that is labeled zero. As shown in Figure 9, assign the coordinate zero to the other end point of the arrow.

We now have two "parallel" number lines and the graph of one ordered pair of a relation. Let us pause and raise a few questions before we go on with our discussion. What is this ordered pair? What does this ordered pair tell us about the experiment? On which of these number lines should we graph the elements of the domain and on which should we graph the elements of the range? Should these number lines have identifying labels and if so, what should they be? This last question necessarily raises the question of what coordinates should be assigned to the points graphed on these number lines. We should also stop to consider whether or not this particular relation is a function. What is the domain and what is the range of this relation?

The ordered pair already graphed is (0,0). The first element of this ordered pair tells us that the number of glugs of water that are in the bottle and the second element is related to the depth of the water in the bottle. The arrow drawn on the graph identifies the line to the left as the one on which we will graph the domain, while the line to the right is the one on which we will graph the range.

On the line to the left lay off a uniform scale. Points on this line 1 cm apart give an appropriate scale for this experiment. Assign the coordinate of each of these points on this line and label the line "number of glugs".

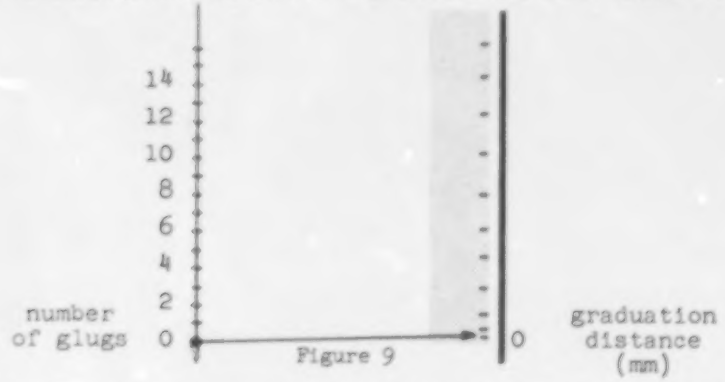


Figure 9

Let us label the other line "graduation distance". In assigning numbers to the points which we marked as we performed the act of filling the bottle,

let us agree to measure this distance accurate to the nearest millimeter. So directly below the label, "graduation distance", write in parentheses (mm). With the aid of a ruler gradusted in millimeters, assign the appropriate coordinates to each mark you made on the tape. Be sure that each coordinate is the distance in millimeters of that particular mark from the zero point.

Now we are ready to draw the arrows that indicate the graph of the ordered pairs which were determined by the experiment. After we have done this, we should stop to consider whether this relation is a function. It appears that the domain of the relation is the set of all volumes of water which can be placed in the bottle and the range is the distance, measured on the side of the bottle, that the water level is from the bottom of the bottle. To each volume of water there is only one depth and therefore only one ordered pair. This relation is indeed a function.

How many ordered pairs did you display in your graph which you made for this experiment? At this point it might be wise to list all of these ordered pairs. As soon as you have done this, look them over carefully. Does your list look something like this

$$\{(0,0), (1,14), (2,25), (3,28), (4,32), (5,40)\}?$$

Surely this is the listing of a set of ordered pairs. But is it the set of ordered pairs which defines the function we are referring to? You will recall that the function was defined as a set of ordered pairs. Our only problem, then, is to decide whether this is the set or only part of the set. Closer examination should assure us that there do exist ordered pairs which we did not graph. For example, we could have found a volume which was half a glug and filled the bottle in half glug volumes rather than glugs. This would have given us more ordered pairs, but the relation would still be the same relation. We are still referring to the volume of water placed in the bottle and the mark made on the tape. We could carry this even farther and determine just how many medicine droppers of water it will take to just fill the pill bottle. From this you could proceed to fill the bottle by adding water, a medicine dropper full at a time.

Since we now agree that the set of ordered pairs we have collected is not the complete set, then we must agree that the graph we have drawn is not the graph of this particular function. As the size of the glug becomes smaller, more glugs are needed to fill the bottle. Figure 10a is a graph of ordered pairs when 40 glugs are needed to fill the bottle.

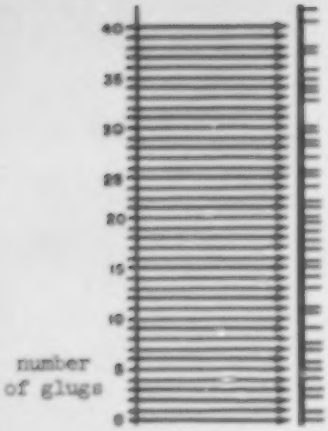


Figure 10a

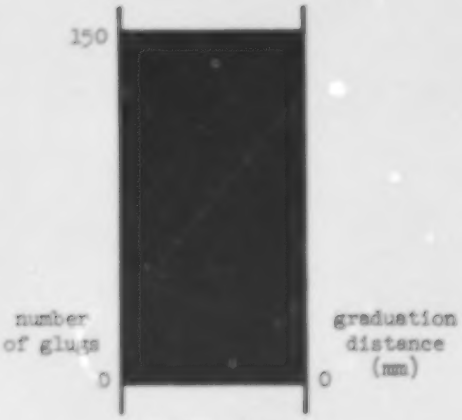


Figure 10b

If a still smaller glug is used, such as a medicine dropper, the graph would look something like Figure 10b.

What information can you read from such a graph? We could probably all agree that there is very little that we could conclude from this graph. Our method of graphing worked very well as long as we had relatively few ordered pairs to graph, but as soon as we extended the number of ordered pairs we wished to graph, this method became cumbersome and the individual ordered pairs had a tendency to lose their identity.

In the next section, we will develop another method of graphing functions which is more generally useful.

3.8 A Coordinate System in a Plane

In the last section we talked about the idea of a coordinate system on a line. Let us pause and review for a moment what we mean by a coordinate system on a line. A coordinate system on a line is determined by any pair of points on it. One point of this pair determines the origin and the other determines the unit-point. The number zero is designated as the coordinate of the origin and the number one is designated as the coordinate of the unit-point.

When we have indicated the coordinates of the origin and the unit-point you will recall that every positive number is associated with a point of the line which is on the same side of the origin as the unit-point. Every negative number corresponds to a point on the opposite half of the line. In this

way the coordinate we have assigned to a point tells us two things. It tells us the distance from the origin to the point, and it also tells us the direction from the origin to the point.

We have already said that number lines can be drawn in different directions. This time let us draw one of the number lines so that it is perpendicular to the other number line. We will call this pair of number lines a rectangular coordinate system. It is not necessary that the two number lines be perpendicular to each other, but this is the type of coordinate system in a plane which we are most likely to see and use.

We will take the intersection of these two lines as the origin of the coordinate systems of both lines. Each number line is called an axis. Often the axis which extends across the paper is called the horizontal axis and the other axis is called the vertical axis. Usually the horizontal axis is named the "x-axis" and the vertical axis is named the "y-axis". The plane determined by these two axes is called the coordinate plane. Let us agree to place the unit-point on the horizontal axis to the right of the origin and the unit-point on the vertical axis above the origin. Coordinates may now be assigned to all points on each axis (Figure 11).

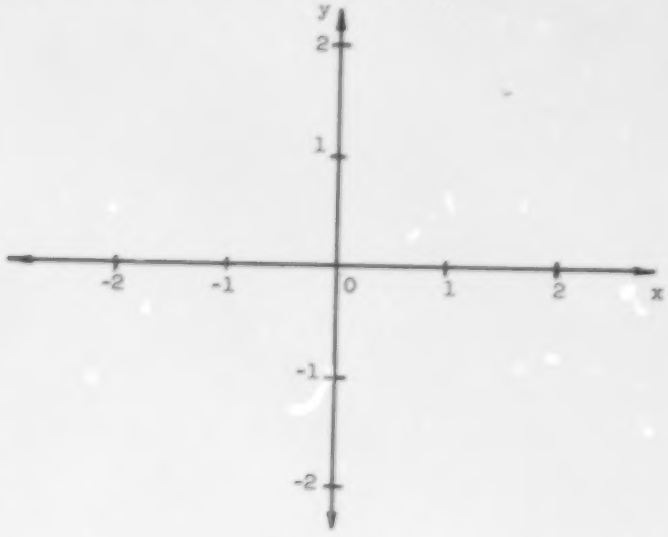


Figure 11

Because of the way these axes are usually shown in pictures on a chalkboard, it is customary to call lines parallel to the horizontal axis horizontal lines, and lines parallel to the vertical axis vertical lines.

We are now ready to define a coordinate system in the coordinate plane.

Consider a particular point first, such as Q in Figure 12, and suppose that the vertical line through Q cuts the horizontal axis in the point whose coordinate is 3.

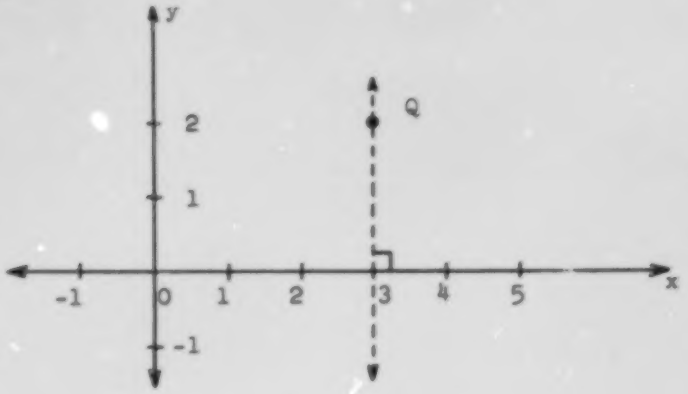


Figure 12

Let us also suppose that the horizontal line through Q cuts the vertical axis in the point whose vertical coordinate is 2 (Figure 13).

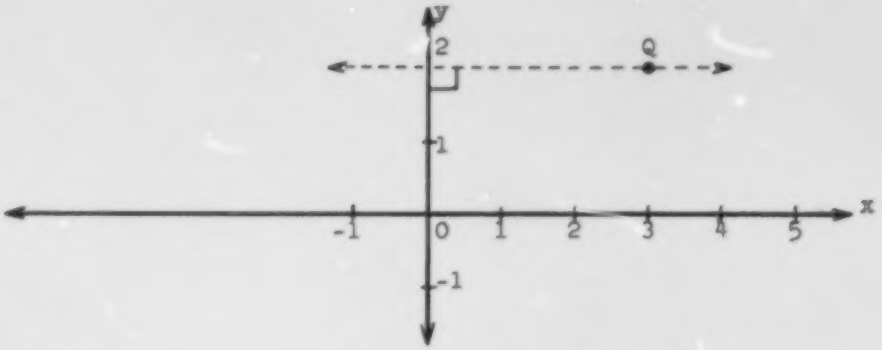


Figure 13

We say, in this case, that the horizontal coordinate of Q is 3, that the vertical coordinate of Q is 2, and that the coordinates of Q are the ordered pair (3,2). The point (3,2) is shown in Figure 14.

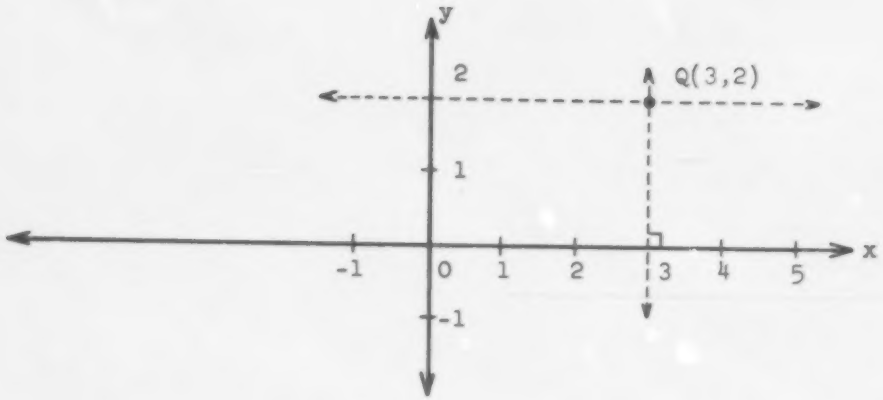


Figure 14

We are now nearly ready for the general case. Let P be any point in the coordinate plane. By the method we just discussed, the point P has a horizontal coordinate and a vertical coordinate, which we refer to as the coordinates of P. The coordinates of P are considered to be an ordered pair of real numbers in which the horizontal coordinate is the first number of the pair and the vertical coordinate is the second.

If we consider the vertical line which we draw through the point Q, we note that the horizontal coordinate of every point on this line is 3. In fact, we might refer to this line as a set of ordered pairs whose first element is 3 and whose second element is a coordinate number. Sometimes we see this last sentence written in symbols as follows: $\{(x,y): x = 3$ and y is any coordinate number$\}$. In this notation the colon, ":", is read "such that" and the sentence is read, "The set of ordered pairs x, y such that x is 3 and y is any coordinate number."

Similarly, the set of ordered pairs whose first element is any coordinate number and whose second element is the number 2 would be the horizontal line which we drew through the point Q.

Suppose we now wish to graph the ordered pair of numbers (5,1) as a point on the coordinate plane. We would first consider the set of ordered pairs whose first element is 5. This would be a vertical line which cuts the horizontal axis at the point whose coordinate is 5 (Figure 15).

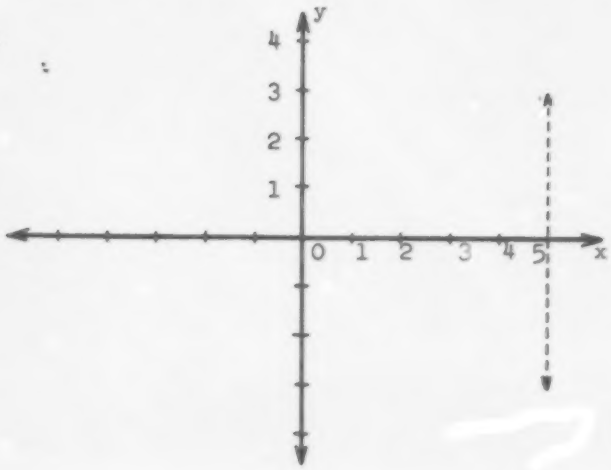


Figure 15

Then we would consider the set of ordered pairs of numbers whose second element is 1. This would be the horizontal line which cuts the vertical axis at the point whose coordinate is 1 (Figure 16).

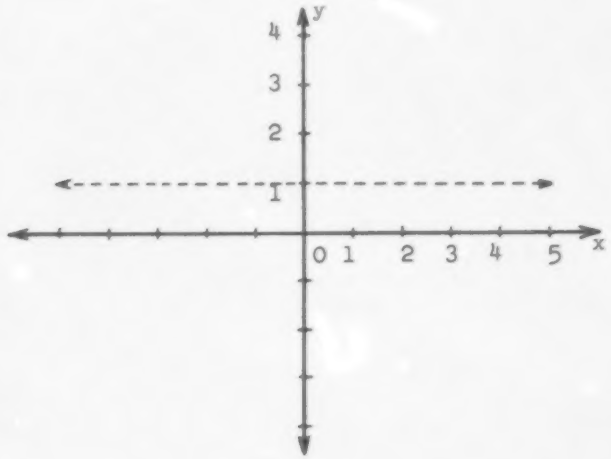


Figure 16

The intersection of the horizontal line and the vertical line is the point whose coordinates are the ordered pair (5,1), as shown in Figure 17.

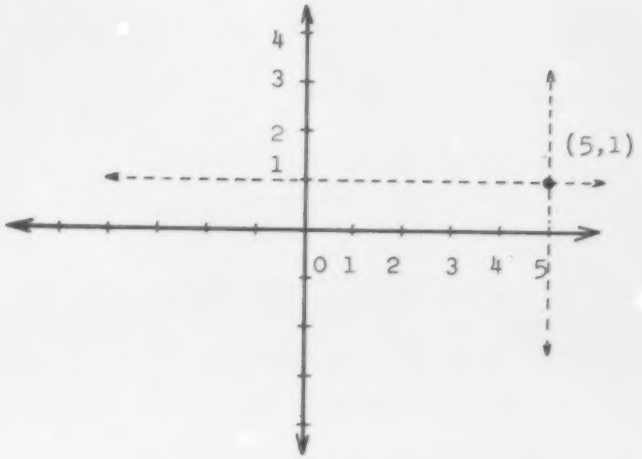
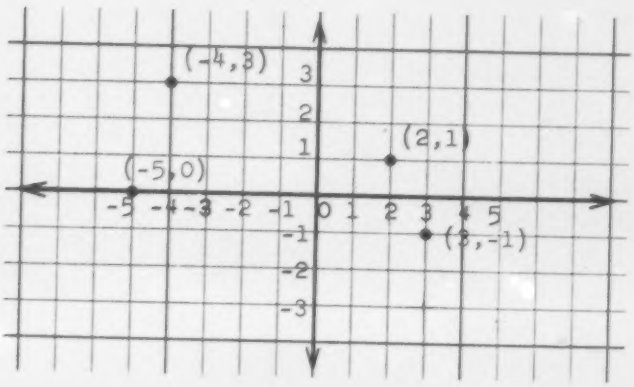


Figure 17

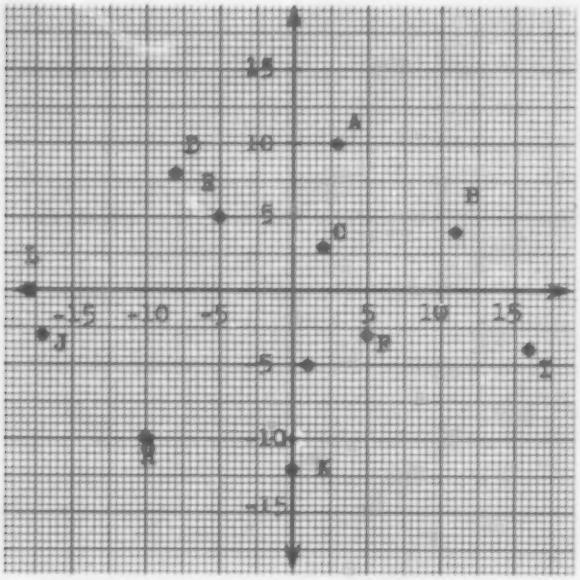
The horizontal line and the vertical line are rarely drawn on the graph. Usually the person graphing the point visualizes these lines in his mind and places a dot on the coordinate plane at the point where the two lines intersect. In order to be clear, the graph should be labeled with the ordered pair which indicates the coordinates of the point.

Example: Plot on a coordinate plane the following set of points: {(2,1), (3,-1), (-5,0), (-4,3)}.



Exercise 4

1. On squared graph paper draw a pair of axes and label them. Indicate the coordinate system on each axis.
   (a) Sketch (with a straight edge) a line which represents the set of points whose horizontal coordinate is 5.
   (b) On the same coordinate plane sketch a line which represents the set of points whose vertical coordinate is 5.
   (c) How many points do these two sets have in common?
   (d) Write as an ordered pair the coordinates of every point of intersection of the sets graphed in (a) and (b).
2. Repeat Problem 1 for the set of points whose horizontal coordinate is 3 and the set of points whose vertical coordinate is 8.
3. Repeat Problem 1 for the set of points whose horizontal coordinate is 8 and whose vertical coordinate is 3.
4. Is the point of intersection of the two sets in Problem 2 the same point as the point of intersection of the two sets in Problem 3? Why?
5. (a) Plot on a coordinate plane the following set of points:
   {(0,0), (-1,0), (1,0), (-2,0), (2,0), (-3,0), (3,0)}
   (b) Do all the points in this set seem to lie on the same line?
   (c) What do you notice about the vertical coordinate for each of the points?
6. (a) Plot the points in the following set:
   {(0,0), (0,-1), (0,1), (0,-2), (0,2), (0,-3), (0,3)}
   (b) Do all the points named in this set seem to be on the same line?
   (c) What do you notice about the horizontal coordinate for each of the points?
7. (a) Plot the points in the following set:
   {(0,8), (1,6), (2,4), (3,2), (4,0)}
   (b) Do all the points named in this set seem to lie on the same line?
8. Write the coordinates of each point in the following graph as an ordered pair:



3.9 Quadrants

In describing the location of a point in the coordinate plane, it is convenient to specify the portion of the plane in which it lies. The horizontal axis and the vertical axis divide the plane into four regions. Each of these regions is called a quadrant. The first quadrant is the set of all points whose horizontal and vertical coordinates are both positive. The second quadrant is the set of all points whose horizontal coordinate is negative and whose vertical coordinates are positive. The third quadrant is the set of all points whose horizontal coordinate and vertical coordinate are both negative. The fourth quadrant is the set of all points whose horizontal coordinate is positive and whose vertical coordinate is negative. We denote these quadrants by I, II, III, IV, as shown in Figure 18.

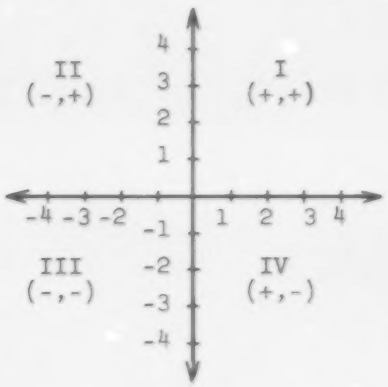


Figure 18

If both of the coordinates are zero, the point is the origin. If the horizontal coordinate is zero and the vertical coordinate is positive, we say that the point is on the positive vertical axis, but if the vertical coordinate is negative, the point is on the negative vertical axis. In a similar manner, if the horizontal coordinate is positive and the vertical coordinate is zero, the point is on the positive horizontal axis; with horizontal coordinate negative, vertical coordinate zero tells us that the point is on the negative horizontal axis.

Exercise 5

1. Given the following ordered pairs of numbers, write the number of the quadrant or the position on an axis in which you find the point represented by each of these ordered pairs:

(a)	(3,5)	(g)	(-3,-1)	(m)	(2,-4)
(b)	(-5,1)	(h)	(7,-1)	(n)	(5,2)
(c)	(1,-4)	(i)	(8,6)	(o)	(-3,0)
(d)	(-4,4)	(j)	(3,-2)	(p)	(-4,-5)
(e)	(0,0)	(k)	(-3,-5)	(q)	(-1,2)
(f)	(0,5)	(l)	(-1,3)	(r)	(3,-1)

3.10 Graphing an Experiment

Let us go back and look at the two experiments we performed earlier in this chapter.

You will recall that when we balanced the books on the edge of the table, it was suggested that we might collect the following set of ordered pairs:

$$\{(1,12), (2,18), (3,22), (4,25), (5,27)\}.$$

At the time, we did not classify this relation as a function. In this particular case we have 5 elements in the domain and 5 ordered pairs. This tells us that each element in the domain appears in only one ordered pair. Another way of saying this would be that there is only one maximum overhang for a given stack of books balanced on the edge of the table. By definition, this relation is a function.

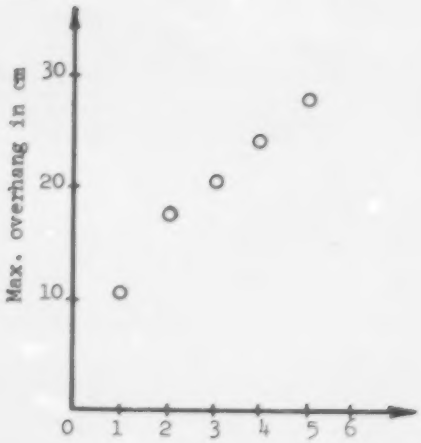
Similarly, in the irregular bottle experiment we obtained a collection of ordered pairs which make up a function.

Let us graph each of these functions on a coordinate plane and see what

we can conclude from these two graphs.

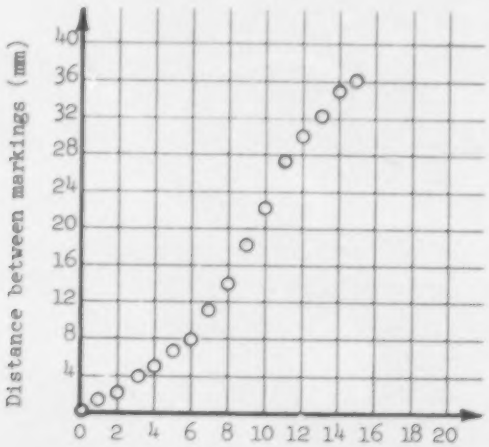
In both cases the elements in the domain will contain no negative numbers. It is certainly ridiculous to balance a negative one book or pour a negative two glugs of water into the empty bottle. It should also be obvious that the range of each of these functions will contain no negative numbers. This tells us that the functions will be limited to the first quadrant, therefore when we draw the axes for each graph there is no need to show the other three quadrants.

The graph of each collection of ordered pairs might look something like this:



Number books in stack

Domain: {1,2,3,4,5}



Number of Glugs

Domain: {0 and 15 and all numbers between 0 and 15}

Figure 19

You will notice that in each case we were careful to indicate the scale being used on each axis. It isn't always necessary to use a unit-length on the horizontal axis which is equal in length to the unit-length on the vertical axis, but care should be taken not to become confused when these different unit-lengths are used. The horizontal axis and the vertical axis should always be labeled to indicate the domain and the range of the function.

Plot the data you obtained for the irregular bottle experiment on a rectangular coordinate system. Notice that the number of glugs are plotted along the horizontal axis with even spacing. You may use any convenient unit

of length on this axis.

Can you see any relationship between the shape of the bottle used and the shape of the graph formed? What do you think was the shape of the bottle used to get the data plotted in Figure 19?

The boy who did the experiment made a mistake and made one of the marks on the tape slightly away from the right position. By looking at the graph of Figure 19 can you see at what point the mistake was made?

Now we should ask if each of the graphs is a complete pictorial representation of that particular function. Remembering that we have defined a function as a particular set of ordered pairs, our problem is simply this: "Does the graph indicate all the ordered pairs of the function and only the ordered pairs of the function?"

The graph of the "book balancing" function is complete. There were only five books in the set of books. At no time did we perform an intermediate task of balancing a fractional part of a book. So in this case the domain of this function is the set which indicates the number of books in each stack $\{1, 2, 3, 4, 5\}$.

The ordered pairs collected in the irregular bottle experiment defined a function. It is possible to find other ordered pairs that have the same relation. This could be accomplished by using parts of glugs.

We should be able to see that the problem of actually measuring extremely small amounts of water and the effect these would have on the depth of the water in the bottle becomes impossible to determine. Our technique of adding water and measuring distances just isn't that accurate, and we certainly don't want to take the time to attempt this kind of measuring if it isn't necessary. Is it possible, then, for the graph to tell us about ordered pairs in the set which were not the result of direct measuring? In other words, can we use the graph to make reasonable predictions of other ordered pairs in this set which defines the function?

Let us pause for a moment and think carefully about the domain of this function. Is there such a volume as $\frac{1}{2}$ glug or .9632 glugs? We do not ask if we measured these volumes but merely ask if such volumes exist. Suppose you decided that the capacity of the bottle was 5 glugs. This indicates, then, that the domain of this function would be the numbers 0 and 5 and all numbers between 0 and 5 which could be used as a measure of volume. Let us again look at the graph of the ordered pairs collected in this experiment. This time, however, let us show the graph of the domain on the horizontal

axis. We can accomplish this by placing a dot at the 0 coordinate and a dot at 5 coordinate and connecting these dots with a solid straight line along the horizontal axis.

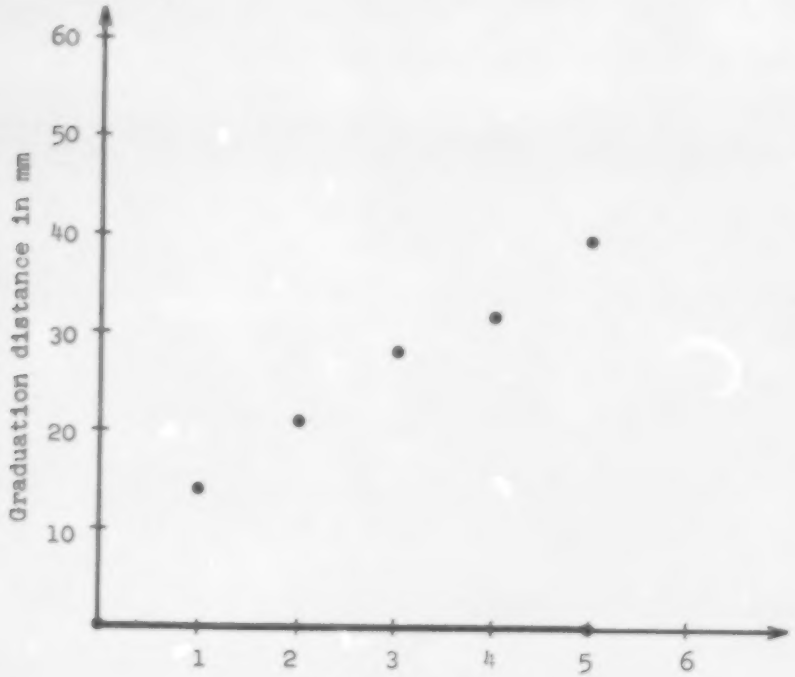


Figure 20

This solid line segment along the horizontal axis clearly shows the domain of the function. By definition of a function, we show that every number in the domain must be associated with a unique number in the range. Clearly, our graph does not show all of these ordered pairs, and we may not be able to predict other ordered pairs. However, we might again think of our observations as we filled the bottle with water. As we added the water, the depth increased. At no time did the addition of any volume of water cause the depth to decrease. At the same time, though, the depth increased gradually or smoothly. The addition of small amounts of water did not cause great jumps in the depth of the water. These observations should lead us to predict that the points we have graphed should be connected by some kind of line. This line should probably be a smooth curve ⌒ rather than a series of connected straight line segments ⩘ . Also, these ordered pairs on the line should have the property that as the elements in the domain get larger, the corresponding elements in the range also get larger until it reaches a maximum value of 40. From this we can see that the range of the function is the number 0, the number 40 and all numbers between 0 and 40.

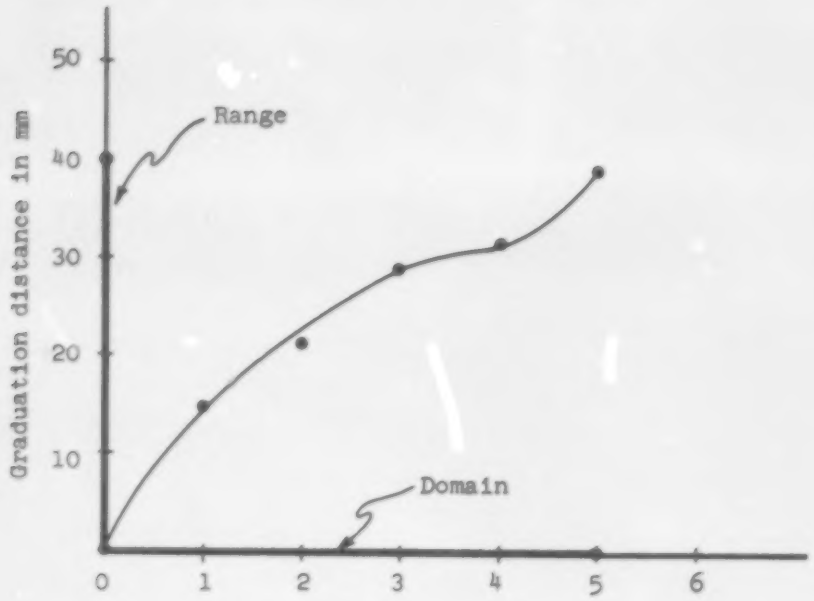


Figure 21

We realize, of course, that the points on the curve, other than those points we actually plotted, are really only a prediction of what the graduation depth would be if we were to measure that particular amount of water and place it in the bottle. Taking more measures and plotting these would give us a better basis upon which to make the prediction of the actual shape of the graph.

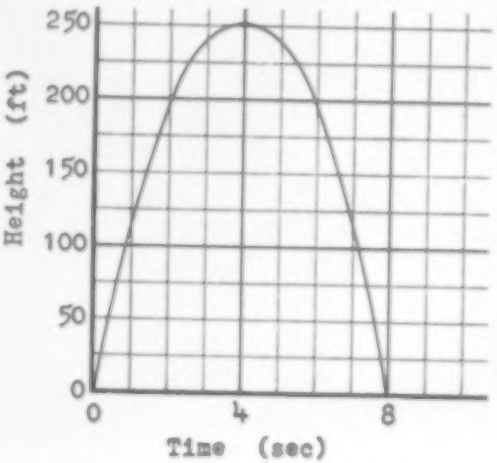
Exercise 6

1. In an experiment a solid material was heated over a burner. The temperature was recorded as a function of the time it took the material to reach a definite temperature. Let the domain be the set of all times from the beginning of the experiment to 18 minutes later and the range be the set of all temperatures from 20°C to 200°C.
   (a) Drew a horizontal and vertical axis and place the appropriate labels for the domain and range on these axes.
   (b) Mark the domain on the horizontal axis with a heavy dark line as in Figure 21.
   (c) Mark the range on the vertical axis.
   (d) Plot the following ordered pairs which were collected while doing the experiment: (0,20), (2,40), (4,60), (6,80), (8,81), (10,82), (12,83), (14,120), (16,160), (18,200).
   (e) Connect these points with a "smooth" curve.

(f) What temperature would you predict for the material at the following times: 1 minute, 5 minutes, 9 minutes, 17 minutes?

(g) At what time would you predict the material would have the following temperatures: $70^{\circ}C$, $100^{\circ}C$, $150^{\circ}C$, $180^{\circ}C$?

2. The following graph was drawn from information gathered in an experiment dealing with a ball thrown into the air. The height of the ball above the ground was plotted as a function of the time it took the ball to reach a definite height.



(a) What is the domain of this function?

(b) What is the range of this function?

(c) How high would you predict the ball would be after $\frac{1}{2}$ second?

(d) How long had the ball been in upward flight when it reached a height of 125 ft?

(e) How long had the ball been in flight when it descended to a height of 125 ft?

(f) Can anything meaningful be said concerning the height of the ball after 10 seconds? Explain.

3.11 Summary

In this chapter we introduced the idea of using ordered pairs to show the relationship between the elements of two sets. A set of ordered pairs is called a relation. The set of all first elements of these ordered pairs is called the domain of the relation. The set of all second elements is called the range of the relation.

When no element of the domain corresponds to more than one element of the range, the relation is called a function.

The graph of a relation helps us to see the information contained in a relation. A rectangular coordinate system is commonly used to provide the framework for graphing a relation. The horizontal axis corresponds to the domain and the vertical axis to the range. The graph of a relation will be used a great deal in the next chapter.

Chapter 4

THE LINEAR FUNCTION

4.1 Graphing Linear Functions Through the Origin

For 15¢ you can buy three packs of gum. So six packs would cost 30¢. Naturally you would expect three times that many packs to cost three times as much. The cost is proportional to the number of packs purchased. In a similar way the distance a baseball travels is proportional to the force with which it is hit. The harder the batter hits the ball the farther it goes. Here is a third example of this simple kind of relation. The height of a stack of two-by-fours is proportional to the number in the stack. Stack height and number of pieces increase or decrease in the same ratio.

Many functions exist in which a change in one quantity causes the other quantity to change in the same ratio. Triple the number of sacks of cement on a scale and the weight triples. Allow a faucet to run only half as long and only half the amount of water flows out.

Not all functions are of this simple kind. Think, for instance, about height and age. Are your height and age changing in the same ratio? When you are three times as old will you be three times as tall as you are now? Fortunately, the ratio of height and age does not stay the same.

In this section we are going to limit our study to those things that change in the same ratio, like weight and the number of identical items weighed. We shall start with an experiment because it will reveal what the graph of this kind of relation looks like. What are the quantities we will graph? They are the lengths of two objects, a new piece of chalk and an unsharpened pencil. Both of these lengths will be measured in a variety of units. The measurements will give a set of ordered pairs. Finally, we will graph this function to see what it looks like.

The lengths of the chalk and pencil should be measured in at least ten different units. You undoubtedly think of the inch and the centimeter first. Measure the lengths of the chalk and the pencil in inches and record this ordered pair in Table 1. Now measure the length of the chalk and the length of the pencil in centimeters. Record this ordered pair in the table.

The rest of the length measures of the chalk and pencil will be made with scales you do not ordinarily think of as rulers. A piece of lined loose

leaf or notebook paper is such a length scale. Lay the piece of chalk on a sheet of lined paper with one end of the chalk on a line. Count the number of spaces spanned by the chalk on the lined paper. Estimate the nearest tenth of the final space. Use the same sheet of lined paper to find the length of the pencil. Record this ordered pair in the table.

A variety of other graduated objects are listed in the "measuring device" column of the table. Use as many of these as are available to find the length of the chalk and the length of the pencil. You should add graduated objects of your own discovery to this list so that you have at least ten measuring devices. Look around the house for objects or in magazines and newspapers for pictures that have equally spaced marks on them. Anything with equal divisions can serve as a length measuring scale. Every measurement should be made to the nearest tenth of a scale division.

Measuring device	Length measure of the chalk	Length measure of the pencil
Ruler (inches)		
Ruler (centimeters)		
Ruler (feet)		
Ruler (meter)		
Notebook paper		
Ordinary graph paper		
Thermometer		
Crossword puzzle		
Graduated cylinder		
Lines on a printed page		
Bead chain (from a bath-tub stopper)		

Table 1

Let's plot these ordered pairs as points on a graph. The domain will be the length measures of the chalk. The range will be the length measures of the pencil.

When all the points have been plotted lay a ruler so that its edge is on the two points that are farthest apart. Where are the eight in-between points? The regularity of these points is evident, isn't it? Did you expect them to line up that way?

If one of your points is off of the line of the others, go back and

remeasure the chalk and pencil using that unit again. You will probably find that you made an error the first time. There's another feature of this graph worth noting. Align your ruler with the points again. Do you have to shift the ruler much to get it on the (0,0) point? Do you think the origin ought to be on line with the experimental points? Observe that the measuring device with the largest unit gave the point closest to the origin. The measures of the lengths of both the chalk and the pencil decrease as the measuring unit increases. The larger the unit the smaller the measure. If you were to measure the chalk and pencil in miles, for example, both length measures would be very close to zero. The experimental points should be in line with the origin.

A set of points are linearly related if they lie on a straight line. Since all the measurements you've made give ordered pairs whose points lie on a straight line, draw a line through them. This line passes through many points in addition to the ten or so you've gotten experimentally. What about all the other points on the line? Does every one of them belong to the relation? If they do belong, we call the relation a linear function. If they do not belong, we call the relation a discrete function in which the ordered pairs are linearly related. By the proper choice of length unit, any one of the ordered pairs could be made to appear in the table. Every point does represent the lengths of the chalk and pencil in some unit. Consequently, an unbroken line should be drawn through the experimental points.

To emphasize that a continuous line should be drawn to represent the function, let's think about an example of a function that is represented by a set of points through which no line should be drawn. The ordered pairs, consisting of the number of two-inch cubes in a stack and the corresponding height of the stack, is a good example. Here's the graph.

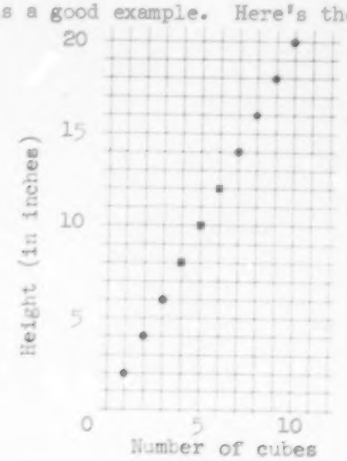


Figure 1

The height increases by two inches for each cube added to the stack. Stack heights of 1, 3, and 5 inches, for example, can never occur. This is called a discrete function because the domain includes only the integers and the range includes only the even integers. The points should not be joined by a line even though they lie on a straight line.

Let's see if there is a linear relation between the measures of lengths measured in feet and in inches. Measure the length of the objects listed in the following table in feet and in inches. Record these ordered pairs in the table.

	inch measure	foot measure
Length of the desk		
Height of the chalk trough		
Length of this page		
Width of the door		

Table 2

Now plot these ordered pairs. Foot measure should be graphed on the horizontal axis and inch measure on the vertical axis. Can you align your ruler with all these points? Should the origin be included? (That's the same as asking: does zero inches = zero feet?) Is the graph of the relation between inch and foot measures continuous? Is the function linear? If so draw a straight line through the points.

In case you need more practice on discovering linear relations, fill in the blanks in the next minute-second function table. Here's the way to do it, based on the first time interval shown in the table. It takes two minutes to fall asleep. Two minutes equals $2 \times 60 = 120$ seconds. Write 120 in the number of seconds column.

Time interval	minutes measure	seconds measure
Time to fall asleep	2	
Time to soft boil an egg	3	
Time to run a mile		240
Time for a TV commercial		90
Length of recess period	5	

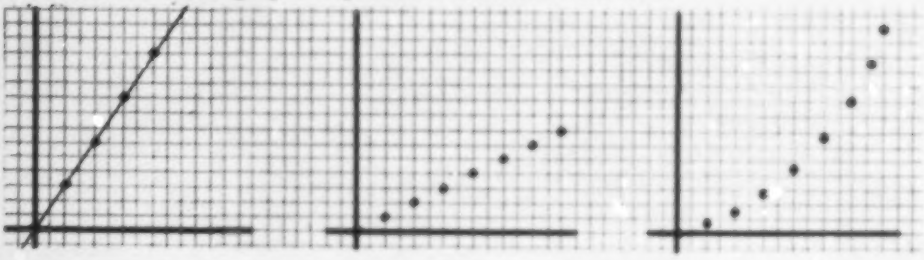
Table 3

Plot these ordered pairs. Number of minutes should be on the horizontal axis. Does the function include the pair (0,0)? That's the same as asking, does

zero minutes equal zero seconds? Is the function continuous? Is the function linear? If so draw a straight line through the points.

Exercise 1

1. Which of these graphs represents a linear relation?



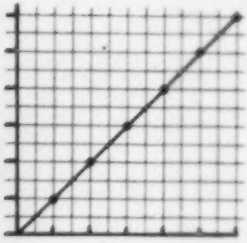
(a) (b) (c)

2. The table shows the corresponding lengths of the side and perimeter of a set of squares.

   (a) Supply the numbers missing from the table.

   (b) Graph the data to see if the relation is linear.

Square	Measure of a side	Measure of the perimeter
A	1	4
B	3	
C	10	
D		60
E		22
F	12.5	

3. 

The graph is that of a linear function. Complete this table from the graph.

B	A
	4
10	
5	
0	
	5

4. Is the function shown in this table linear?

x	y
1	4
2	8
4	16
$\frac{1}{2}$	2
0	0

4.2 Representing Linear Functions by Sentences

A further study of the linear relations which we have already graphed will show a way to recognize this kind of linear function without graphing it. First look at the ordered pairs in the minute-second data table. (See Table 3.) The measure of any time interval in minutes is smaller than the corresponding measure in seconds. The number connecting any ordered pair in this table is 60. Multiply any number of minutes by 60 and you get the corresponding number of seconds. This fact can be written in sentence form:

Seconds measure = 60 × Minutes measure.

Would a similar statement describe the relation between the measure of any length in feet and corresponding measure of the length in inches? Is there some number by which you can multiply feet or inches to get the equivalent value of the other measure? Which of the following is it?

number of feet = 12 × number of inches

number of inches = 12 × number of feet

The same type of sentence can be written to describe the relation between the measures of the length of a piece of chalk and a pencil. These measures were recorded in Table 1. You can easily find the number connecting the two elements of any ordered pair in this table. To do so divide any length measure of the pencil by the corresponding length measure of the chalk. (The quotient should be between 2 and 3 since the pencil is a little over twice as long as the chalk.) Record this quotient in the blank column of Table 4.

Measuring device	Pencil length ÷ chalk length
Ruler (inches)	
Ruler (centimeters)	
Ruler (feet)	
Ruler (meter)	
Notebook paper	
Ordinary graph paper	
Thermometer	
Crossword puzzle	
Graduated cylinder	
Lines on a printed page	
Bead chain	

Table 4

Find the quotient for all the other ordered pairs. Record each in its proper place in the quotient column. Add all these quotients and divide the sum by the number of quotients. This average is the number the chalk length must be multiplied by to get the pencil length. This is a third function that can be written in sentence form.

Pencil length = Some number × Chalk length

Do you see how similar the verbal statements of these three linear functions are?

Seconds measure = 60 × Minutes measure

Inch measure = 12 × Foot measure

Pencil length = Some number × Chalk length

All these linear functions can be stated in the same way:

The measure of one quantity	=	Some number	×	The measure of the other quantity.

Whenever a function can be described by a sentence of this form it is linear. Its graph is a straight line through the origin.

Exercise 2

1. Fill in the missing numbers:
   (a) Number of feet = _____ × Number of miles.
   There are _____ feet in 3 miles.
   (b) Number of quarts = _____ × Number of gallons.
   There are _____ quarts in 8 gallons.
   (c) Number of hours = _____ × Number of days.
   There are _____ hours in $\frac{1}{4}$ of a day.
   (d) Number of ounces = _____ × Number of pounds.
   There are _____ ounces in 20 pounds.
   (e) Diameter of a circle = _____ × radius.

2. Write the relation between:
   (a) Pounds measure and the corresponding tons measure.
   (b) Foot measure and the corresponding yard measure.
   (c) Hours measure and the corresponding minutes measure.
   (d) Cubic foot measure and the corresponding cubic yard measure.
   (e) Year measure and corresponding day measure.
   (f) The circumference and the corresponding diameter of a circle.

3. A meter is longer than a foot. The number relating them is approximately 3.28. How would you write this relation?

4. A gallon is a smaller volume than a cubic foot. The number relating corresponding measures is approximately $7\frac{1}{2}$. Write the relation between a number of gallons and the corresponding number of cubic feet.

5. In the preceding four problems thirteen linear functions are described. Can you give additional examples of linear functions?

4.3 Functions of the Form: $y = mx$

All the linear relations we have studied can be written in similar form. Here are a few of them.

(a) Number of minutes = 60 × Number of hours

(b) Number of inches = 36 × Number of yards

(c) Circumference = π × Diameter

These statements look even simpler when letters are used instead of words. In the place of:

Number of minutes write m
Number of hours write h
Number of inches write i
Number of yards write y
Circumference write c
Diameter write d

and the three statements become:

(a) $m = 60\ h$

(b) $i = 36\ y$

(c) $c = \pi\ d$

When the statements are written in this brief, symbolic way they are called equations. Note how much alike these equations are. Each contains two letters and one number. Notice that the equations are true for the sets of ordered pairs as follows:

(a) (h,m)
(b) (y,i)
(c) (d,c)

where the letters in the ordered pairs have the same meanings as above.

Convert these sentences into equations:

(a) Centimeter measure = 100 × Meter measure

(b) Pint measure = 8 × Gallon measure

(c) An automobile is traveling at a speed of 50 mi/hr.
Number of miles traveled = 50 × Number of hours.

All have the form

$$y = mx.$$

This is the equation of a linear function whose graph is a straight line through the origin. For example:

let $m = 60\ h$

$i = 36\ y$

$c = \pi\ d$

The identical arrangement of the two letters and one number in these linear functions suggests a further simplification:

where y stands for m, i or c

x stands for h, y or d

m stands for 60, 36 or π

Exercise 3

1. The linear relation:

Number of minutes = 60 × Number of hours

becomes $m = 60\ h$

when written as an equation. Use this equation to complete the following statements:

(a) If $h = \frac{1}{4}$ hours, $m =$ _____ minutes.

(b) If $h = \frac{1}{2}$ hour, $m =$ _____ minutes.

(c) If $m = 300$ minutes, $h =$ _____ hours.

(d) If $m = 20$ minutes, $h =$ _____ hours.

2. A car averages 20 miles on a gallon of gas. Write the equation relating number of miles traveled to number of gallons of gas used.

3. Complete the table using this linear relation:

$y = 4x$.

x	y
3	
$\frac{1}{2}$	
	12
	22
2.5	

4. U.S. paper money is available in bills of the following denominations: \$1, \$2, \$5, \$10, \$20, \$50, and \$100.

(a) Make a table showing the equivalent number of quarters of each of these bills.

(b) Graph this set of ordered pairs.

(c) Is this graph continuous?

(d) Write an equation relating the value of any bill in dollars and the equivalent number of quarters.

4.4 Slope

Previous sections have considered linear functions whose graphs are straight lines through the origin. We discovered that such functions can be represented by an equation of the form $y = mx$. Here's a review of these ideas in terms of an example.

(a) Knowing that 60 seconds = 1 minute, we can write the equation:

Number of seconds = 60 × Number of minutes

$$s = 60\,m$$

(b) Use the above equation to complete the table.

m	s
0	0
	90
	120
3	
4	

(c) Finally, note that the completed table gives this graph.

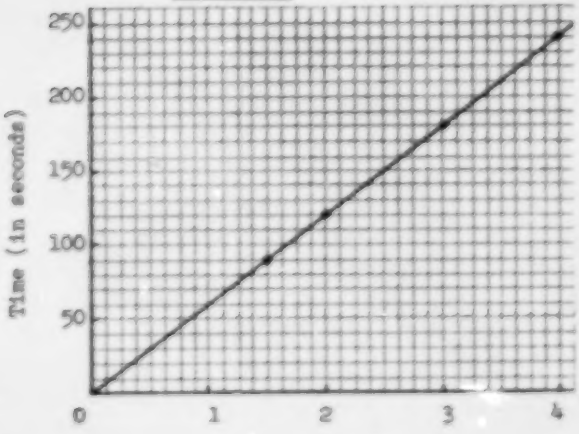


Figure 2

Take note of the order of these three steps:

(a) We started with an equation.

(b) Made a table of ordered pairs.

(c) Plotted these pairs and drew the line.

It is often quite useful to do these steps in a different order. For example, when a scientist performs an experiment, he does the following:

(a) He makes measurements which give him a table of ordered pairs.

(b) He graphs these pairs and draws a line through the points.

(c) Finally he figures out the equation of the function from the graph.

Here are these two procedures, side by side, so you can compare them.

(a) Equation	(a) Table
(b) Table	(b) Graph
(c) Graph	(c) Equation

We will now learn how a scientist goes from the graph to its equation.

All linear functions whose graphs are straight lines through the origin have an equation of the form:

$$y = mx.$$

These examples will refresh your memory:

(a) Diameter of a circle = 2 × radius	$d = 2r$
(b) Number of feet = 3 × Number of yards	$f = 3y$
(c) Number of quarts = 4 × Number of gallons	$q = 4g$
(d) In general	$y = mx$

All of these equations look the same except for the value of m. The graph of each equation is a straight line through the origin as shown in Figure 3.

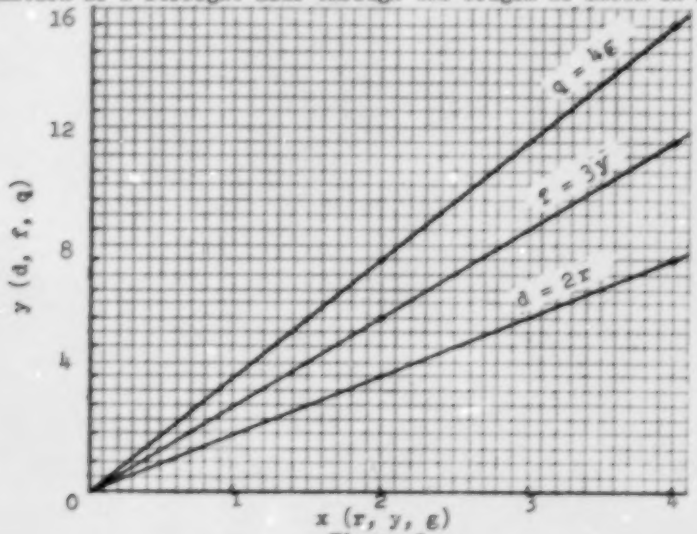


Figure 3

These linear functions differ in two ways:

(a) The equations have different values for m.

(b) Each graph of these linear functions has a different rise from left to right.

NOTE: The larger the value of m, the steeper the rise of the line. Since m controls the steepness of the line it is called the <u>slope</u> of the line.

Let's use the graph of the equation $q = 4g$ to discover the method for finding the value of the slope, m.

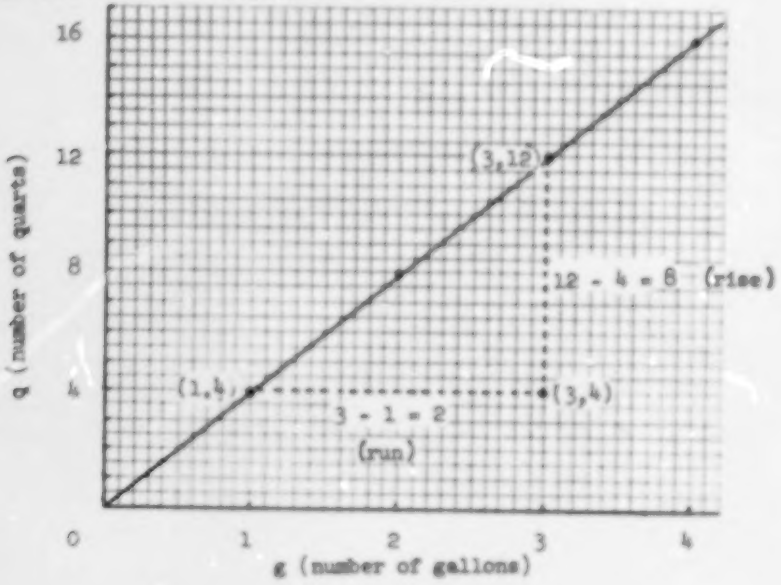


Figure 4

The equation of this line is $q = 4g$ so its slope, m, is 4. Here's how to get the slope, 4, from the graph.

(a) Select <u>any</u> two points on the line. Let's use (1,4) and (3,12).

(b) Draw a horizontal line (shown dotted) from point (1,4) to the right.

(c) Draw a vertical line (also dotted) from (3,12) down to meet the horizontal dotted line. They meet at the point (3,4).

(d) Find the difference, called the <u>rise</u>, between the ends of the vertical dotted line. This difference is 12 - 4 = 8.

(e) Find the difference, called the <u>run</u>, between the ends of the horizontal dotted line. This difference is 3 - 1 = 2.

(f) The slope, m, has the value:

$$m = \frac{\text{rise}}{\text{run}} = \frac{\text{vertical difference}}{\text{horizontal difference}} = \frac{12 - 4}{3 - 1} = 4.$$

It's worthwhile repeating that the slope, m, can be found from <u>any</u> two points on the line. We have just used (1,4) and (3,12) to get m = 4. Let's find the slope, 4, using the points (0,0) and (4,16):

(a) Drew a horisontal line from (0,0) to the right.

(b) Drew a vertical line from (4,16) down. These two lines should meet at the point (4,0).

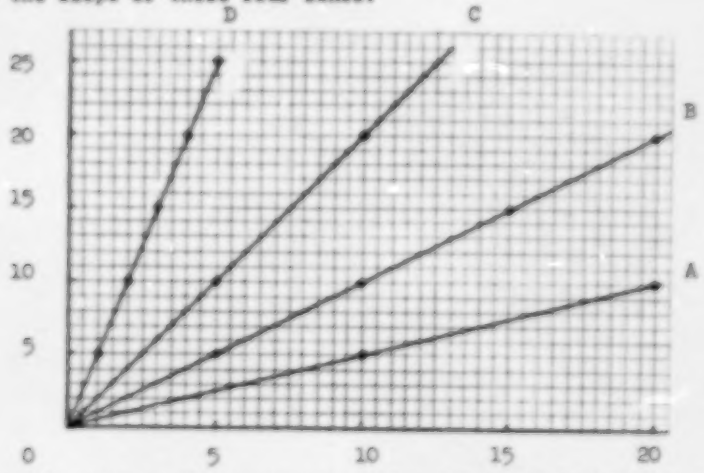
(c) Find the rise and run.

(d) Do you agree that $m = \frac{16 - 0}{4 - 0} = 4$?

Exercise 4

1. Use the points (3,12) and (4,16) to find the slope of the line graphed on page 98.

2. Find the slope of these four lines.



3. The graph relating the measures of time intervals in seconds and minutes is shown in Figure 2. What is the slope of this graph?

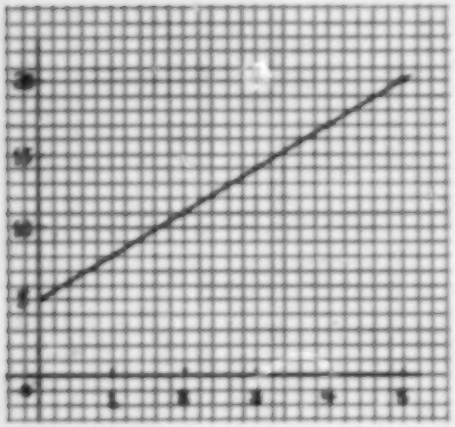
4. If you were to graph the following equations, what would be the slopes of the lines?

   (a) $d = 365y$

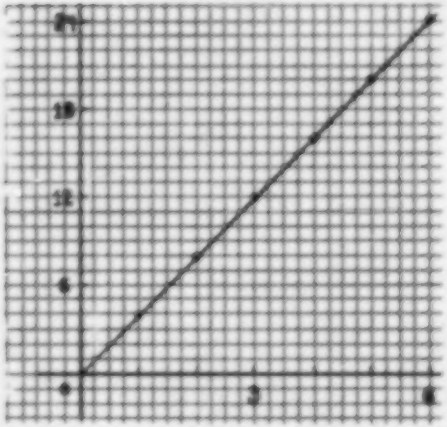
   (b) $q = 16p$

   (c) $y = 3x$

5. Find the slope of this line, using the method outlined on page 98.



6. Drew a line that has twice the slope of the line shown on the graph. Drew another line with half the slope.



7. Try to drew the graph of the equation $y = 5x$, using only the fact that the slope is 5.

4.5 Coat Hanger Experiment

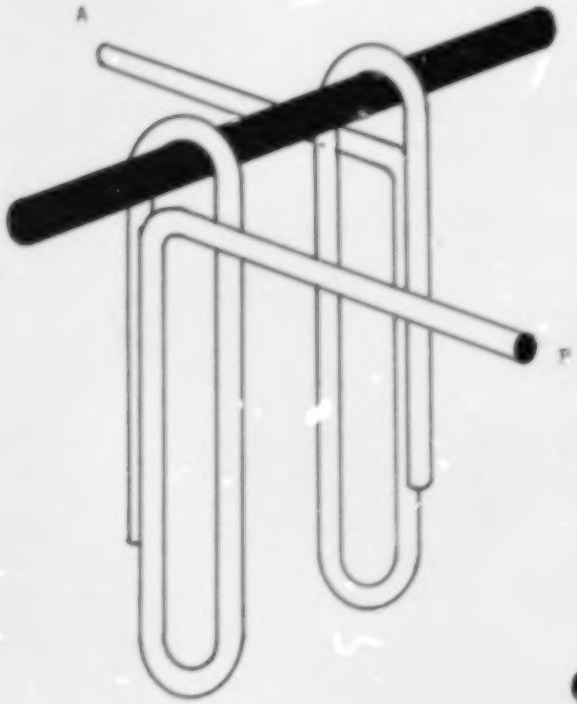
In the previous section you learned that a scientist frequently finds it useful to start with a function table, make a graph and from it figure out the equation relating the variables. He often does an experiment (as you did when you measured the length of a piece of chalk and a pencil) which gives him a data table. He graphs these data. Finally he uses the graph to figure out the relation between the variables. Let's see if we can carry out such a process.

First get the general idea of the experiment. When you walk out on the end of a diving board it bends down. Suppose you were joined by a friend who weighs the same as you. The board will bend more. How much more? You're tempted to answer, twice the weight, therefore twice the bend. But can you be absolutely certain without making measurements? This is an example of the kind of experiment we are going to do. Bridges sag under a load of automobiles; the top of the Empire State Building sways several feet in a high wind; a bow bends noticeably when the archer pulls back on the string to shoot an arrow, etc. Everything bends when a force is exerted on it. Perhaps

the amount is slight, even unnoticeable to the eye, but it is always there.

The purpose of the experiment is to discover the relation between the amount a coat hanger bends and the load hanging on it. Follow these detailed instructions to do the experiment.

Attech an unsharpened pencil with cellophane tape to the top of the desk so that about one-third of the pencil extends beyond the edge. Lay several books on top of the pencil to keep the tape from pulling loose.



COAT HANGER

Figure 5a
Paper clips are separated to indicate position of pointers A and B.

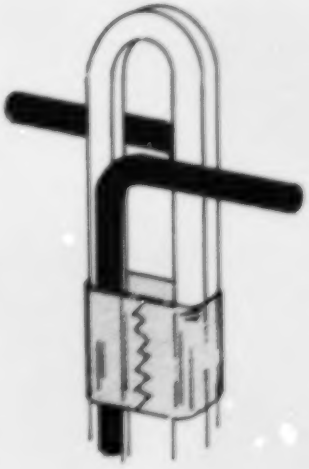


Figure 5 b
Pointers have been shaded and clips have been joined by tape.

Figure 5

Study Figure 5 carefully before you attempt the next step. Bend up the inside loop of two jumbo size paper clips to form pointers. (The pointer should be perpendicular to the body of the clip.) Slip the two clips on the bottom side of the hanger. The pointers should extend in opposite directions. Tape the clips together to form a single hook for the masses.

Hook the hanger on the pencil. Slide the paper clips to the exact center of the hanger. Tape a ruler to the side of the desk, in such a position that the paper clip pointer is aimed at some centimeter mark on the ruler. With a set of masses at hand you are ready to begin the experiment.

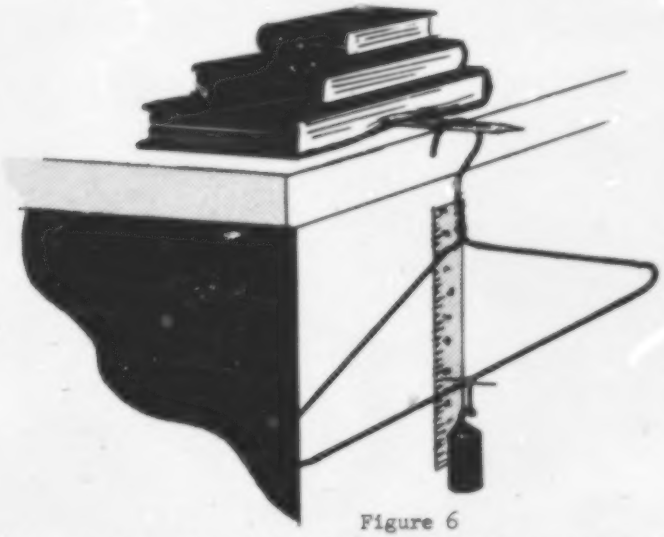


Figure 6

Hang a 100 gram mass on the clip and measure, to the nearest millimeter, the amount the hanger bends down. Record the mass and the amount of bend it caused in the data table.

M (mass in gm)	B (bend in mm)
0	0
100	
200	
etc.	

Table 4

Increase the mass to 200 grams and measure the <u>total</u> deflection (bend) it produces. Record this pair of numbers. Continue to add to the mass, 100 grams at a time, until you've reached 1000 grams. For each new load measure the corresponding total bend and record this ordered pair in the table.

The complete data table is a set of ordered pairs showing how the bend in a coat hanger is related to the load on it. Another way to display this function is to graph the ordered pairs. If the graph turns out to be a straight line through the origin, the function can be written as an equation of the form:

$$B = mL$$

where B is the bend, L the load and m the slope.

Make a graph of these data. Plot load on the horizontal axis and bend on the vertical axis. If the points approximate a straight line draw the best straight line you can through them.

Determine the slope, m, of the line. Put this value of the slope into the equation

$$B = mL$$

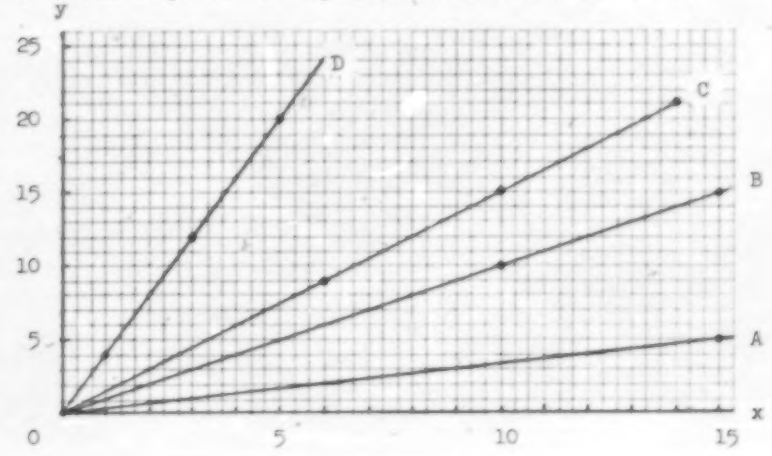
and you have the equation of the line.

Exercise 5

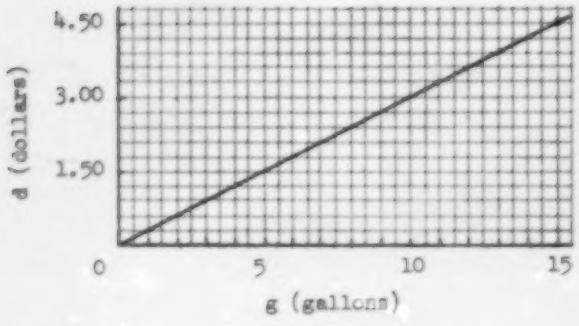
1. Suppose there is a linear relation between the amount a diving board bends and the load on it. Then it would be like the coat hanger.
   (a) If the board bends down 1.5 inches when you (120 pounds) get on it, what would be the total bend when your friend, who also weighs 120 pounds, joins you?
   (b) Suppose you exert a force of 400 pounds on the board when you jump on it. How much will it bend?

2. Make a data table and a graph of the linear function described in Problem 1. What is the slope of this line?

3. What is the slope and the equation of each of these lines?



4. A gas station attendant could use the following graph to figure out how much to charge a customer for the gas he puts in his tank.



(a) What is the slope of the line?

(b) What is the cost of 8 gallons of gas?

(c) What is the cost of 6.5 gallons?

(d) Write an equation that could be used to figure gasoline bills.

4.6 Graphing Linear Functions in General - Spring Experiment

Do you think you could lift an object that weighs as much as you do? Some people can do it. They have built up their strength gradually through muscle developing exercises. A popular arm strength developer you probably have seen is a set of springs with a handle at each end.

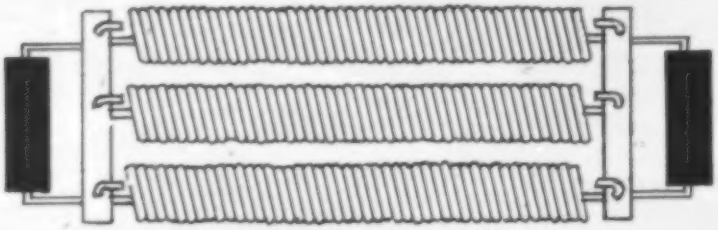


Figure 7

You can hold one handle in each hand and pull in opposite directions or you can put a foot through one handle and pull up on the other with one or both hands. Why can your dad stretch the springs farther than you can? The reason is no secret; he's stronger. What causes a spring to stretch? What determines how much it stretches? Do you think there's any relation between the pull on a spring and its length? Let's try to find out. Tape a pencil to the desk. Weight the pencil with a couple of books to keep the tape from pulling loose. Slip one end of the spring your teacher has provided over the

pencil. Let the spring hang vertically so that the free end can stretch downward. Pull down on the free end with a weak force. Now pull harder. Was there any change in the length of the spring? Does it become twice as long for twice the pull? Of course you couldn't find out the relation between the length and pull by this simple experiment because you don't know how much you changed the pull when you went from a weak to a stronger pull.

To discover the exact relationship between the length of a spring and the pull on it a more careful experiment must be done. Instead of pulling down with your hand, hang an object of known mass on the spring. If you start with a 200 gram mass and then replace it by a 400 gram mass you will have exactly doubled the pull on the spring. Do you think the length of the spring with a 400 gram mass hanging on it will be twice the length for a 200 gram mass? First hang a 200 gram mass on the spring and measure its length. Here's a diagram that may help you understand how to make the measurements.

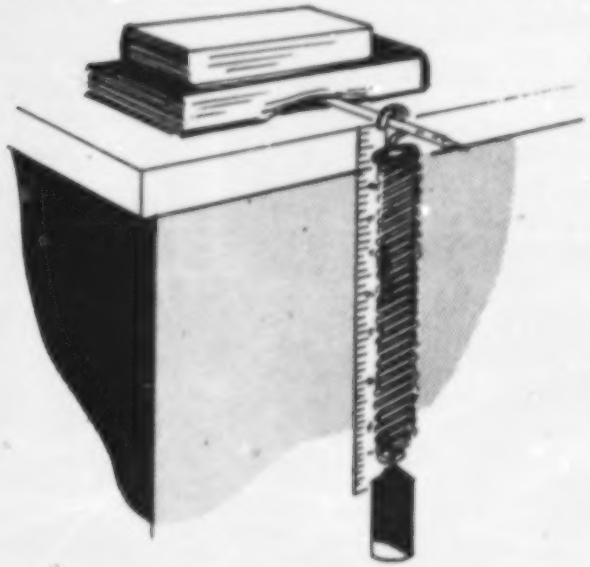


Figure 8

So far you have one set of measurements, a mass of 200 grams and the corresponding length of the spring. Make a data table like the following and write this ordered pair in it.

Number of grams	Number of centimeters
200	

Table 5

Now double the mass hanging on the spring and measure the length of the spring. Record this pair of numbers in the data table. You doubled the weight pulling on the spring. Did the length of the spring double? Clearly the relation between the length of the spring and the mass hanging on it is still unknown.

Perhaps the equation for the relation can be found from a graph. To make a graph more pairs of mass-length numbers are needed. Add another 100 grams to the mass hanging on the spring and measure the length. Record the ordered pair in the data chart. Continue to add masses, 100 grams at a time, until the spring has stretched to about three times its normal length. Every time you add 100 grams, measure the length of the spring and enter this length and the corresponding mass in the table.

Make a graph of your data, plotting the measures of the mass on the horizontal axis and the measures of the spring length on the vertical axis. Draw the best line you can through the points.

This graph differs, in one striking way, from the other straight lines we have found so far. It does not pass through the origin. This indicates that the spring length does not change in the same ratio as the mass hanging on it. Of course, you discovered this fact at the start of the experiment when you doubled the mass and found that the spring's length didn't double. Now you see how the graph shows it by failing to pass through the origin.

Let's see if we can find the equation of this line even though it doesn't pass through the origin. Suppose we consider an equation of the form:

$$L = mM$$

where L = spring length and M = mass stretching the spring. Find the slope of the spring graph and put it into this equation. Now complete the following table of ordered pairs from this equation.

M (grams)	L (centimeters)
	0
	200
	400
	600
	etc.

Table 6

Plot these points on the same graph paper you used for the spring experiment. Draw a straight line through the points. Evidently L = mM is not the equation

of the line gotten from the spring experiment. The best we can say for L = mM is that it is the equation of a line through the origin that runs in the same direction, or, in other words, has the same slope as, the "spring" line.

Next, we will show how to get the equation of a line, not through the origin, when you know its slope. The line whose equation we seek is the solid line in this diagram.

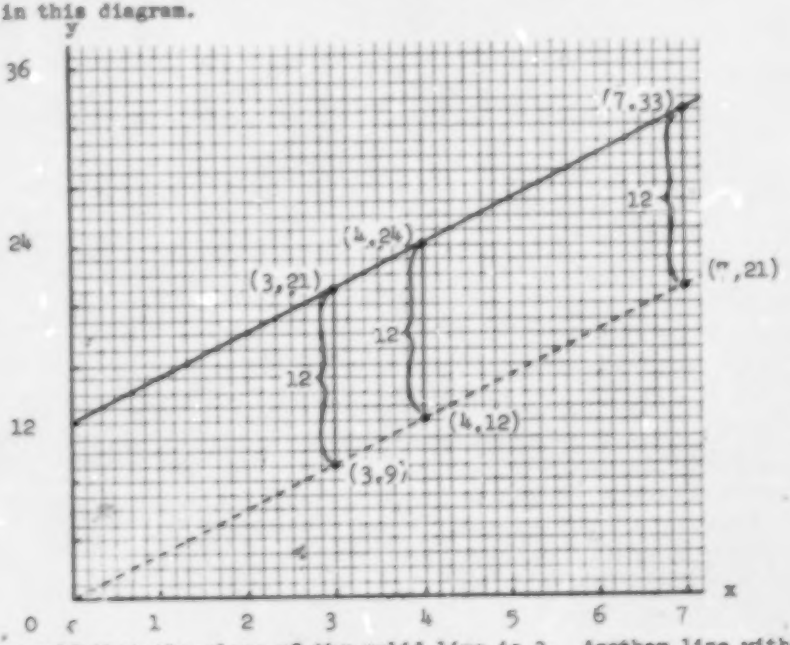


Figure 9

Prove to yourself that the slope of the solid line is 3. Another line with a slope of 3 has been drawn. It's the dotted line through the origin. Do you agree that the equation of this dotted line is:

$$y = 3x.$$

Watch how this equation can be used to get points <u>on the solid line</u>.

(a) When x, in y = 3x, is replaced by a number, say 3, y's value is 9. Hence (3,9) is a point on the dotted line. If we add 12 to 9 (12 is the vertical distance between the solid and dotted lines) the result is (3,21), a point <u>on the solid line</u>.

(b) Try this for another value of x, say 7. For x = 7, y = 21. Add 12 to 21 and we get (7,33), a point on the solid line.

(c) Here's a third trial. For x = 4, y = 12. Add 12 and the result is (4,24), a point on the solid line.

In general the y-coordinate of a point of the solid line is obtained by multiplying the x-coordinate by the slope, 3, and then adding 12. Points

are found on the

dotted line from the equation $y = 3x$,

solid line from the equation $y = 3x + 12$.

Therefore, $y = 3x + 12$ is the equation of the solid line. The 3 is easy to find; it's the slope of the line. The 12 is also easy to find; it's the y-coordinate of the point where the line crosses the y-axis. This coordinate is called the y-intercept. Its symbol is b. Hence the equation of a straight line can be written in the form:

$$y = mx + b.$$

Now you can write the equation of the spring graph. Find the value of the slope, m, and the value of the y-intercept, b, and put them into the formula

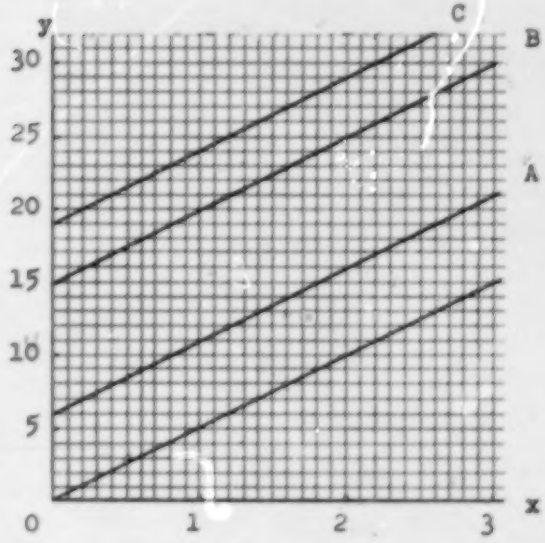
$$y = mx + b.$$

Exercise 6

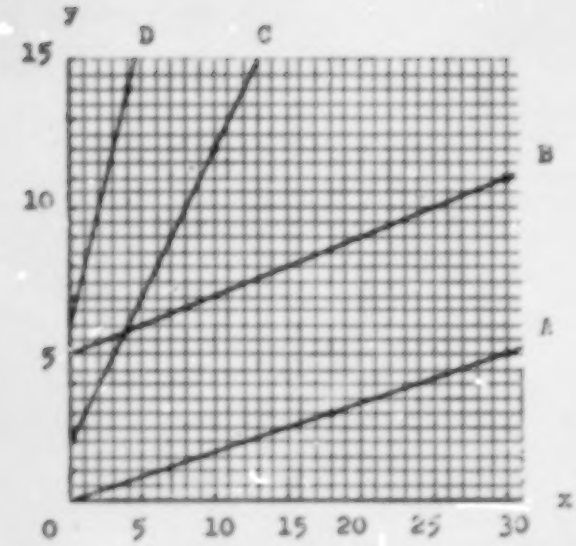
1. In the figure to the right, the four lines each have the same slope. The line through the origin has the equation

   $y = 5x$.

   Write the equation of each of the other three lines.



2.



Use the accompanying figure to fill in this chart.

Line	m	b	Equation $y = mx + b$
A			
B			
C			
D			

3. The equation of a line is $y = 2x + 6$. Graph this line.

4. The equation of a line is $y = 6x + 1$. Find out which of the following points are on this line without graphing it:

   $(1,7)$ $(3,19)$ $(2,10)$ $(\frac{1}{2},4)$ $(\frac{1}{3},3)$.

4.7 The Centigrade-Fahrenheit Experiment

One encyclopedia lists several dozen different units in which length can be measured. You have been using a few of them -- the inch, foot, yard and mile -- from early childhood. In the early lessons of this course you became familiar with the meter, centimeter and millimeter. A variety of time measuring units are likewise in common use -- the second, minute, hour, etc. Angles are measured in at least two units. You have probably measured angles in degrees. You may also have heard of the unit of angle measurement called the radian. All these examples indicate the great variety of standard units available for measuring any property. This makes it important to be able to shift from a measurement in one unit to the same measurement in a different unit. You've done this automatically for years with some length units:

3 feet equals 36 inches; two gallons equals eight quarts; $\frac{1}{2}$ hour equals 30 minutes. Today we are going to discover another relationship between two units. They are the degree Centigrade and the degree Fahrenheit. These two units measure temperature changes.

Suppose a visitor from Europe told you that he liked to go swimming whenever the temperature reached twenty-five degrees. You might be astonished until you realize that he is not using the Fahrenheit scale you are familiar with. Seventy-seven degrees Fahrenheit is the same temperature as twenty-five degrees Centigrade. The next experiment will reveal the mathematical relation that can be used to convert from Fahrenheit to Centigrade and vice versa.

Here is a brief description of the experiment. Corresponding Centigrade and Fahrenheit temperature measurements are made of six quantities of water whose temperatures range from very cold to hot. These ordered pairs of numbers are graphed. The equation relation F to C is determined from the graph.

Now do the experiment by following these detailed instructions. Half fill a container of about one quart capacity with crushed ice. Pour in just enough water to cover the ice. Stir the slush thoroughly with both thermometers. When you think the temperature of the water has fallen to that of the ice, read both thermometers. (Any time you make a temperature measurement, position the bottom of the thermometer about one inch below the surface of the liquid.)

Record the Centigrade and Fahrenheit temperatures in the data table opposite "Pure ice water".

	Number of ^{o}C	Number of ^{o}F
Pure ice water		
Salty ice water		
Cold tap water		
Cool water		
Lukewarm water		
Hot water		

Table 7

By this time a quantity of the ice in the container will have melted. Pour off the excess water. For the next temperature measurements we will want just enough water to fill the spaces between the chunks of ice. No water should show above the ice. Now pour 20 heaping tablespoons of salt into the ice water and stir for several minutes to dissolve most of the salt. Put both thermometers into the salty ice water and take both readings. (Don't forget about the one inch depth for the thermometers.) Enter the set of numbers in the data table. Is the salty ice water temperature higher or lower than the pure ice water temperature? By the way, what is the lowest temperature you can read with your Centigrade thermometer? How would you write this number which is below zero to distinguish it from the same number of degrees above zero? Have you written the Centigrade temperature of salty ice water correctly? Now let's get some temperatures above freezing. Measure the C and F temperatures of the coldest water you can get out of the faucet. Record this pair of numbers. Get a pair of temperatures between ice water and cold tap water by mixing equal amounts of ice water and cold tap water. Get the highest pairs of temperatures by measuring the C and F temperatures of the hottest water you can get from the faucet. One more pair of numbers will complete the function table. Mix equal amounts of the hottest and coldest tap water available. Measure the C and F temperatures and record the numbers.

If you have made all of the measurements suggested above, you have six pairs in your function table. The next step is to graph this function. The graph will be most useful if you:

(a) Plot C on the horizontal axis and F on the vertical axis.
(b) Draw the C axis one-third of the way up from the bottom of the graph paper.
(c) Draw the F axis one-third of the way from the left edge.

Plot the data in the table. Draw the best straight line that fits all these points.

This is the first experiment that has given us points outside the first quadrant. The reason is, the Centigrade temperature fell below zero, that is, became negative. How many of your six experimental points have a negative coordinate? What combination of C and F temperatures would give a point in the third quadrant? Would it be possible to have a point in the fourth quadrant?

Exercise 7

1. What is the slope of the C vs F graph?
2. What is the y-intercept of the C vs F graph?
3. Put these numbers in the equation: $F = mC + b$. This is the equation relating the C and F temperature scales.
4. Use this equation to find the F temperature that corresponds to
   (a) 20 degrees C.
   (b) 45 degrees C.
   (c) -5 degrees C.
   (d) 0 degrees C.
5. Plot the points found in Problem 4 on the experimental graph. Do they fall on the C vs F line?
6. (a) Plot the data in the table at the right.
   (b) Find the slope and y-intercept of the line.
   (c) Find the equation of this line.

x	y
-1	3
-2	0
-4	-6

4.8 Summary

Linear functions were first introduced through the graph, second through the sentence and third through the equation of the form $y = mx$. These functions when graphed passed through the origin. When $y = mx$ was introduced many functions were shown to be of the same form and differing only in the value of m. The value of m was recognized first as a conversion factor and then as a constant ratio

$$m = \frac{\text{vertical distance}}{\text{horizontal distance}} = \frac{\text{rise}}{\text{run}}.$$

This ratio was then defined as the slope of the line representing the function and the equation.

The coat hanger experiment provided an example of a set of ordered pairs or function whose graph was a straight line through the origin. Therefore, the equation was of the form $y = mx$.

Then the spring experiment provided an example of a set of ordered pairs or function whose graph was a straight line not through the origin. Therefore, the equation was of the form $y = mx + b$. The Centigrade-Fahrenheit experiment gives another example of this type. It also introduced the need for negative values.

GLOSSARY

Part I

ADDITION PROPERTY OF ORDER -- If a, b and c are real numbers and if $a < b$, then $a + c < b + c$. Also, if $a > b$ then $a + c > b + c$.

APPROXIMATION -- A result that is not exact, but is accurate enough for a particular situation.

AXIS (COORDINATE) -- Any line used to aid in determining the location of points in the plane.

COMPOUND STATEMENT -- A statement constructed from simple statements by use of the connectives "and" or "or".

COORDINATE(S) ON A LINE -- The number associated with a point of the number line is called the coordinate of the point.

COORDINATE(S) ON A PLANE -- The numbers associated -- as an ordered pair -- with a point of the plane are called the coordinates of the point.

COUNTING NUMBERS -- An element of the set {1, 2, 3, 4, 5, ...}. Also called natural numbers.

DEFLECTION -- The amount of bend (as indicated by a pointer relative to a fixed scale).

DIAGONAL -- A straight line segment connecting non-consecutive vertices (going from corner to corner).

DISCRETE -- A discrete set of points refers to a set of points each of which is clearly separated from the others, that is, a set of isolated points. For example, the natural numbers on the number line.

DOMAIN -- The domain is the set of first elements of the ordered pairs in a relation or function.

ELEMENT -- A member of a set.

EQUATION -- An open sentence involving equality.

EXPONENT -- The particular use of a numeral to indicate how many times a certain number should be used as a factor.

FORCE -- Force is a physical concept which can be described loosely as the push or pull on an object.

FUNCTION -- A function is a set of ordered pairs such that each element of the domain appears in one and only one ordered pair.

HORIZONTAL -- Across as opposed to up and down; a straight line following the direction of the horizon.

INEQUALITIES -- A simple sentence with the symbol ">" or "<" as the verb phrase is called an inequality.

INTERCEPT -- The point on a number line at which a second line meets it.

LENGTH -- The measurement of the distance between two points.

LINEAR -- Pertaining to straight lines.

MASS -- Mass is a fundamental property of a body. It is not the same as the weight of the body. On the earth's surface, the weight of an object is proportional to its mass.

MATHEMATICAL MODEL -- A mathematical representation of a physical object or event which can be used to predict information about the object or event.

MATHEMATICAL SENTENCE -- A mathematical statement which is either true or false, but not both.

MAXIMUM VALUE -- The greatest value.

MEASUREMENT -- The process of comparing some object or event with some unit which we have chosen.

NEGATIVE EXPONENT -- In a number such as 10^{-n}, where $n = 1, 2, 3, 4, \ldots$, the -n is a negative exponent. It is interpreted:

$$10^{-n} = \frac{1}{10^n}$$

NEGATIVE REAL NUMBERS -- The set of real numbers associated with points to the left of zero on the number line is the set of negative real numbers.

NONTERMINATING -- Unlimited; not coming to an end; expressed in an infinite number of terms.

NOTATION -- Symbols denoting quantities, operations, or relations.

NUMBER LINE -- When a one-to-one correspondence has been established between the points of a line and the real numbers we call the line a number line.

ORDERED PAIR -- A set containing exactly two elements, (a,b), in which one element is recognized as the first element.

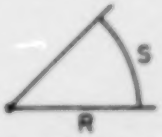
ORIGIN -- The intersection of the horizontal and vertical axes in a coordinate system. This point is represented by the ordered pair (0,0).

PERPENDICULAR LINES -- Two lines which meet at right angles.

PROPORTIONAL -- Two related quantities are said to be proportional if their ratio is always the same.

QUADRANT -- One of the four regions into which the coordinate axes divide the plane. They are usually numbered counter-clockwise.

RADIAN -- One radian is the angle subtended at the center of a circle by an arc equal in length to one radius.



$1 \text{ radian} = \frac{s}{r}$

When s = r, the angle is 1 radian.

RANGE -- The range is the set of second elements of the ordered pairs in a relation or function.

RATIO -- The ratio of a number "a" to a number "b" ($b \neq 0$) is the quotient $\frac{a}{b}$.

REAL NUMBERS -- The set of all numbers associated with points on the number line. A number which can be represented by a finite or infinite decimal expansion.

RELATION -- A relation is a set of ordered pairs. When the pair (x,y) is in the set and we use R to represent the relation, we say that x R y is true.

SCIENTIFIC NOTATION -- The practice followed in mathematics and science of writing numbers as a number between one and ten multiplied by the appropriate power of ten. For example,

$$216 = 2.16 \times 10^{2}$$
$$0.0043 = 4.3 \times 10^{-3}$$

SET -- A well-defined collection of elements.

SIGNIFICANT FIGURE -- If a number is written in scientific notation ($a \times 10^n$), the last digit to the right in "a" is significant. Otherwise, the last non-zero digit to the right in a number is the last significant figure.

SLOPE -- The slope measures the steepness of the inclination of a line. It is the ratio of the rise to the run.

TRANSITIVE PROPERTY -- If a relation R has the property that whenever a R b and b R c are true statements, then a R c is a true statement and we say that R has the transitive property.

UNIQUE -- Just one. Consisting of one and only one. Leading to one and only one solution.

UNIT OF MEASURE -- An object or event of our choice which we compare with the object or event to be measured.

UNIT POINT -- The point associated with the place-value notation referring to the value of the first place. Also, the point on a number line corresponding to the number "1".

VARIABLE -- A symbol which represents a definite though unspecified number from a given set of admissible numbers.

WEIGHT -- The weight of a body is the measure of the force caused by the earth pulling on that body.